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PORTFOLIO OPTIMIZATION: A SURVEY

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Optimization models play an increasingly role in financial decisions. This paper analyzes the portfolio optimization model which is the most important of them. We are discussing the mathematical models and modern optimization techniques for some classes of portfolio optimization problems more important criteria. Portfolio optimization problems are based on mean-variance models for returns and for riskneutral density estimation. The mathematical portfolio optimization problems are the quadratic or linear parametrical programming sometimes with integer variables.

Key words: Markowitz; portfolio optimization; absolute deviation; portfolio diversification; efficient frontier; Sharpe ratio; minimax model; integer variables; fuzzy expected return.

ПОРТФЕЛЬНАЯ ОПТИМИЗАЦИЯ: ОБЗОР

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Показано, что оптимизационные модели играют все более значимую роль в принятии финансовых решений. Анализируются некоторые наиболее важные модели оптимизации инвестиционного портфеля. Обсуждаются современные методы оптимизации для некоторых классов задач с наиболее важными критериями. Отмечено, что классические оптимизационные портфельные задачи базируются на исторических рядах доходностей со среднеквадратическим отклонением от ожидаемой прибыли в качестве меры риска; с математической точки зрения задачи оптимизации портфеля являются задачами квадратичного или линейного программирования, иногда с целочисленными ограничениями.

Ключевые слова: Г. Марковиц; портфельная оптимизация; абсолютное отклонение; диверсификация портфеля; эффективная граница; коэффициент Шарпа; модель минимакса; целочисленные переменные; нечеткая ожидаемая доходность.

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Introduction

Conception of an optimal portfolio of assets was first time mentioned by Louis Bacheliers in his doctoral thesis which was defended in 1900 in Paris. Unfortunately, this thesis exactly like the theory of optimization created by L. Kantorovich and T. Kupmans the Nobel Prize winners in economy were less common among financial managers. They managed to use primary skills of actuarial mathematics, elementary concepts of share fare value (price). The modern portfolio theory was firstly reviewed in the work written by Markowitz [1] and Sharpe [2] who were awarded Nobel Prize in Economics in 1990. This theory is seems to be of high importance. If you make an inquiry about "portfolio theory" and "portfolio optimization" using the search engine *Google.com* you will be given about 2.5 million links for the first one and about 13,8 million links for the second one. Moreover the term "portfolio management" has about 21 million links.

The standard Markowitz portfolio model (model based on Euclidean metric of risk estimation)

Let's suppose that investor has the possibility to choose from the variety of different financial assets like securities, bonds and investment projects. The main point is to define investment portfolio $x = (x_1, \ldots, x_n)$, where x_j is proportion of the asset j. Then the budget constraint is

$$\sum_{j=1}^{n} x_j = 1, \; x_j \ge 0, \; j = \overline{1, n}. \tag{1}$$

It is valuable to say, that absolute weightings of assets could be included in the Markowitz. For instance, by K we denote the investor's initial capital. Then the budget constraint (1) might be replaced for

$$\sum_{j=1}^{n} K_j \bar{x}_j = K, \; x_j \ge 0, \; j = \overline{1, n}, \tag{2}$$

where K_j is the price of asset j. If all assets are infinitely divisible replaced variables

$$x_j = \frac{K_j \bar{x}_j}{K},$$

we get budget constraint (1).

Markowitz's portfolio model [3] assumes to use two criteria: portfolio expected return and portfolio volatility (measure of risk adjusted). Important to add that theory uses the historical parameter, volatility, as a proxy for risk, while return is an expectation on the future.

The return $R(x)$ of the portfolio x is the component-weighted expected the return R_j of the constituent assets. The expected return of an asset is a probability-weighted average of the return in all scenarios. Calling p_t the probability of scenario t and r_{jt} the return in scenario t, we may write the expected return as

$$r_j = E(R_j) = \sum_{t=1}^{T} p_t r_{jt}.$$

It's assumed that all scenarios t (historical) are equal probability in the future, then $p_t = \frac{1}{T}$ and $r_j = \sum_{t=1}^{T} \frac{r_{jt}}{T}$ (see table).

The function of the expected return of the portfolio is needed to be maximized

$$r(x) = E(R(x)) = \sum_{j=1}^{n} x_j r_j \to \max. \tag{3}$$

If we suppose that $r_1 \ge \ldots \ge r_n$ then optimal solution of the problem (1), (3) is $x_{\text{opt}} = (1, 0, \ldots, 0)$, i. e. all capital should invest in the most profitable asset (greedy solution). Clearly, it is very risky. That is why investors add (upper bound constraint) $x_j \le u_j$, $j = \overline{1, n}$, to budget constraints. In this case greedy solution has following form

$$x_{\text{opt}} = \left(u_1, \ldots, u_k, \left(1 - \sum_{j=1}^{k} u_j\right), 0, \ldots, 0\right),$$

where $\sum_{j=1}^{k} u_j \le 1$ and $\sum_{j=1}^{k+1} u_j \ge 1$ and stays optimal. It is possible further to add constraints for diversification of risks. However, Markowitz proposed other approach.

One of the best-known measures of risk is standard deviation of expected returns. Let's σ_{ij} is covariance of the returns i and j, i. e. $\sigma_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - r_i)(r_{jt} - r_j)$.

Markowitz derived the general formula for the standard deviation of the portfolio (risk of the portfolio) as follows:

$$\sigma(x) = \sqrt{E(R(x) - r(x))^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j \to \min. \tag{4}$$

The variance of all asset's returns is the expected value of the squared deviations from the expected return

$$\sigma^2 = \sum_{t=1}^{T} p_t (r_t - E(r))^2.$$

Remark that the covariance matrix $\sigma = (\sigma_{ij})_{n \times n}$ is positively semi-definite and consequently $\sigma(x)$ and $\sigma^2(x)$ are convex functions. That is why standard Markowitz's portfolio model (1)–(4) is bi-criteria optimization problem with linear (3) and convex quadratic (4) objective functions.

In some occasions standard deviation could be substituted for k-order target risk:

$$\sigma(x) = E\left[(R(x) - r(x))^k\right]^{1/k}.$$

Let's apply Markowitz's model to the problem of the optimization portfolio of blue chips, hi-tech corporation's shares, real estate and treasure bonds. The annual times series for the return are given below for each asset between six years.

Portfolio problem with four assets

$j \backslash t$		r_{jt}						$r_j = E(R_j)$
		1	2	3	4	5	6	
Blue chips	x_1	18.24	12.12	15.23	5.26	2.62	10.42	10.648 3
Hi-tech shares	x_2	12.24	19.16	35.07	23.46	–10.62	–7.43	11.98
Real estate market	x_3	8.23	8.96	8.35	9.16	8.05	7.29	8.34
Treasury bonds	x_4	8.12	8.26	8.34	9.01	9.11	8.95	8.631 7

Average annual percentage r_{jt} is specified

$$r_{jt} = \frac{P_{jt+1} - P_{jt}}{P_{jt}},$$

where P_{jt} is asset price j at instant time t.

The return and covariance matrixes can be easily find in the "Mathematica" system by using built-in functions Mean and Covariance. The covariance matrix is

$$\sigma = \begin{pmatrix} 29.0552 & 40.3909 & -0.2879 & -1.9532 \\ 40.3909 & 267.344 & 6.8337 & -3.6970 \\ -0.2879 & 6.8337 & 0.3759 & -0.0566 \\ -1.9532 & -3.6970 & -0.0566 & 0.1597 \end{pmatrix}.$$

The *first* approach leads to the task of minimizing the variance of the portfolio (1) return given a lower bound on the expected portfolio return

$$r(x) \geq k, \tag{5}$$

i. e. under all possible portfolios x, consider only those which satisfy the constraints, in particular those which return at least an expected return of k. Then among those portfolios determine the one with the smallest return variance. Problem (1), (4), (5) is quadratic optimization problem with a positive semi-definite objective matrix σ:

$$\sigma^2(x) = 29.0552x_1^2 + 80.7818x_1x_2 - 0.5758x_1x_3 - 3.9064x_1x_4 + 267.344x_2^2 +$$
$$+ 13.6677x_2x_3 + 7.3940x_2x_4 + 0.3759x_3^2 - 0.1133x_3x_4 + 0.1597x_4^2 \to \min,$$

$$10.6483x_1 + 11.98x_2 + 8.34x_3 + 8.6317x_4 \geq k,$$
$$x_1 + x_2 + x_3 + x_4 = 1, \; x_j \geq 0, \; j = \overline{1, 4}.$$

This problem can be solved by using standard quadratic programming algorithms or in a very efficient way by using the computing system "Mathematica" and it's built-in function Minimize. Setting in the problem (1), (4), (5) for portfolio optimization and solving it for guaranteed return $k = 10.7$ %, we get the optimal portfolio ($x_1 = 0.9523$, $x_2 = 0.0437$, $x_3 = 0$, $x_4 = 0.0040$) with risk $\sigma(x) = 5.4959$ % (one of the corner portfolio).

The *second* approach we consider the task of maximizing the mean of the portfolio return $r(x)$ under a given upper bound k for the variance $\sigma(x)$:

$$\sigma(x) \leq k. \tag{6}$$

Problem (1), (3), (6) is a linear parametric programming with an additional convex quadratic constraint (6) and parameter k.

This problem can be also efficiently solved by using the "Mathematica" system and it's built-in function Maximize. Setting in the problem (1), (2), (6) for portfolio optimization and solving it for as example $k = 1$ %, we get the optimal portfolio ($x_1 = 0.2189$, $x_2 = 0.0114$, $x_3 = 0$, $x_4 = 0.7697$) with return $r(x) = 9.1103$ %.

A portfolio x is efficient (Pareto optimal) if and only if no other feasible portfolio that improves at least one of the two optimization criteria without worsening the other. An efficient portfolio is the portfolio of risky assets that gives the lowest variance of return of all portfolios having the same expected return. Alternatively we may say that an efficient portfolio has the highest expected return of all portfolios having the same variance. The efficient frontier sur-plane (r, σ) is the image $(r(x), \sigma(x))$ of all efficient portfolios x. Let's plot the efficient frontier by using the built-in function ParametricPlot in "Mathematica" system (fig. 1).

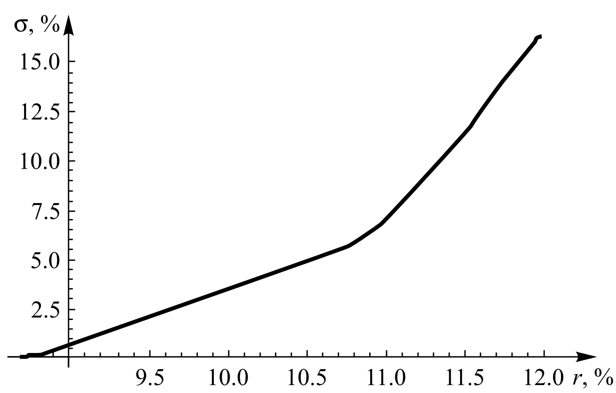

Fig. 1. The efficient frontier

While choosing an efficient portfolio we could apply for weighting objective function approach. The *third* approach is based on using the Carlin theorem of coincidence Pareto-optimal solutions in (1)–(4) in optimal solutions in the one-criterion parametric optimization with parameter k:

$$kr(x) - (1 - k)\sigma(x) \to \max. \tag{7}$$

Here the parameter $k(0 \leq k \leq 1)$ shows investor's risk. This problem can be also easily solved by using built-in function Maximize in the system "Mathematica".

The lower $k = 0$ the less risk we apply for the model, investor is more conservative. Minimal risk is 0.0884 % with portfolio ($x_1 = 0.0537$, $x_2 = 0$, $x_3 = 0.1776$, $x_4 = 0.7687$) and return 8.687 % (another corner portfolio).

If $k = 1$ investor must accept risk in order to receive higher returns. Maximal risk is 16.350 7 % with portfolio ($x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$) and return 11.98 %.

This algorithm for parametric quadratic programming solves the problem (1), (7) for all k in the interval $[0; 1]$. Starting from one point on the efficient portfolio the algorithm computes a sequence of so called corner portfolios $x_{\text{opt}} = (x_{\text{opt}1}, \ldots, x_{\text{opt}m})$. These corner portfolios define all efficient portfolio are convex combinations of the two adjacent corner portfolios: if x'_{opt} and x''_{opt} are adjacent corner portfolios with expected returns

$r\left(x'_{\text{opt}}\right)$ and $r\left(x''_{\text{opt}}\right)$, $r\left(x'_{\text{opt}}\right) \le r\left(x''_{\text{opt}}\right)$ then for every $r\left(x_{\text{opt}}\right) = \lambda \cdot r\left(x'_{\text{opt}}\right) + \left(1 - \lambda\right) \cdot r\left(x''_{\text{opt}}\right)$ the efficient portfolio x_{opt} is calculated as $x_{\text{opt}} = \lambda x'_{\text{opt}} + \left(1 - \lambda\right) x''_{\text{opt}}$, $0 \le \lambda \le 1$.

For instance, find corner portfolios for treasury bonds (x_4) with the portfolio return $k \in \left[8.5, 11.9\right]$ by using built-in function Evaluate in the “Mathematica” system (fig. 2).

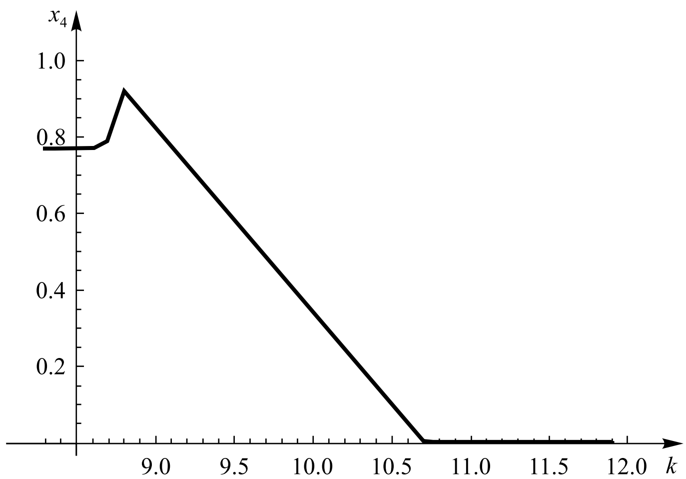

Fig. 2. The corner portfolios for treasury bonds

Corner portfolios for other assets can be find by the same way. There are three corner portfolios: for returns $k_1 = 8.687$ %, $k_2 = 8.8$ % and $k_3 = 10.7$ %. Solving the portfolio optimization problem for return $k = 8.8$ %, get the optimal portfolio ($x_1 = 0.0757$, $x_2 = 0.0051$, $x_3 = 0$, $x_4 = 0.9192$) with risk $\sigma\left(x\right) = 0.1819$ % (the last corner portfolio).

The efficient portfolio x_{opt} is calculated as

$$x_{\text{opt}} = \lambda_1 x_{\text{opt1}} + \lambda_2 x_{\text{opt2}} + \lambda_3 x_{\text{opt3}},$$

where x_{opt1} ($x_1 = 0.0537$, $x_2 = 0$, $x_3 = 0.1776$, $x_4 = 0.7687$), x_{opt2} ($x_1 = 0.0757$, $x_2 = 0.0051$, $x_3 = 0$, $x_4 = 0.9192$), x_{opt3} ($x_1 = 0.9523$, $x_2 = 0.0437$, $x_3 = 0$, $x_4 = 0.0040$) and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $0 \le \lambda_j \le 1$.

Model with risk-free asset (Tobin’s model)

Risk-free asset hypothetically corresponds to be short-term government securities. Conditionally it is assumed that the variation of the government securities return r_0 is equal zero. Considering the following Tobin’s model [4] for portfolio $x = \left(x_0, x_1, \ldots, x_n\right)$ with risk free asset x_0:

$$x_0 + \sum_{j=1}^{n} x_j = 1, \ \ x_j \ge 0, \ \ j = \overline{0, n},$$

$$r\left(x_0, x\right) = r_0 x_0 + r\left(x\right) = r_0 x_0 + \sum_{j=1}^{n} x_j r_j \to \max,$$

$$\sigma\left(x_0, x\right) = \sqrt{x_0^2 \sigma_0^2 + x_p^2 \sigma_p^2 + 2 x_0 x_p \sigma_{0p}} = \sqrt{x_p^2 \sigma_p^2} = x_p \sigma_p = \sigma\left(x\right). \tag{8}$$

Obviously, the expected rates of return on all risky assets are not less asset, i. e. $r_j \ge r_0$.

If we take some definite efficient portfolio, we could figure all portfolios with risk free assets on CML (capital market line) (fig. 3):

$$E\left(R_C\right) = r_0 + \sigma_C \frac{E\left(r_m\right) - r_0}{\sigma_m},$$

where r_m is return of the market portfolio (depending on the market index and its risk is σ_m).

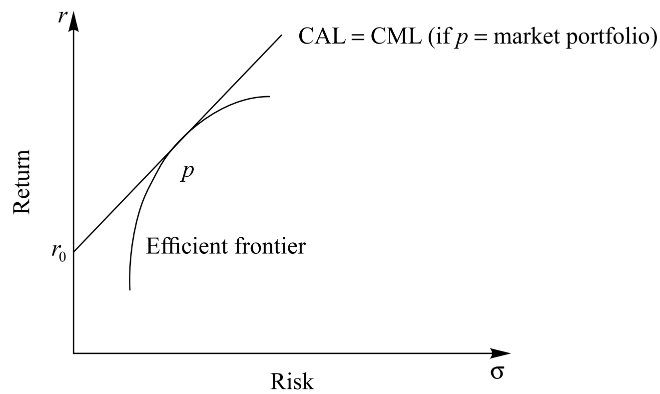

Fig. 3. The capital market line

It is interesting to note, if someone has the possibility to choose not only between the given risk portfolio and risk-free assets but also to choose a structure of the risk portfolio then there exists the unique optimal solution ($x_1 = 0.057\,031\,2$, $x_2 = -0.005\,940\,04$, $x_3 = 0.265\,938$, $x_4 = 0.682\,971$), not depended on investor's risk (solving by the "Mathematica" system).

Multi-objective model for portfolio optimization

The main problem in optimization portfolios is that the portfolios are extremely concentrated on a few assets which are a contradiction to the notion of diversification. Therefore there is scope for introducing another criterion with one for diversification and the best candidate for this. They usually solve quadratic problem for portfolio optimization and then apply entropy measure for infer how much portfolio is diversified. In paper [5] supplement maximize Shannon's entropy and skewness of portfolio:

$$E_n(x) = -\sum_{i=1}^{n} x_j \log x_j \to \max,$$

$$S(x) = \sum_i \sum_j \sum_k \gamma_{ijk} x_i x_j x_k \to \max,$$

where $\gamma_{ijk} = E\left[\left(R_i - r_i\right)\left(R_j - r_j\right)\left(R_k - r_k\right)\right]$ is central third moment of returns.

Model based on Minkowski absolute metric of risk estimation

Konno and Yamazaki [6] propose a linear programming model instead of the quadratic model. Quite widespread to evaluate risk using the Minkowski metric l1 in which deviation is sum of absolute values, i. e. risk l1 of the portfolio return (absolute deviation) is defined as

$$\sigma(x) = E\left\{\left|R - E(R)\right|\right\} = E\left[\left|\sum_{j=1}^{n} r_j x_j - E\left[\sum_{j=1}^{n} r_j x_j\right]\right|\right] = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j\right|.$$

Under the assumption on normal distribution the absolute deviation is equivalent to the standard deviation as the measure of risk [6].

That allow insert additional variables y_t into the model (1), (3) and

$$\sum_{t=1}^{T} y_t \to \min, \tag{9}$$

under the condition

$$y_t + \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \ge 0,\ t = \overline{1, T}, \tag{10}$$

$$y_t - \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \ge 0,\ t = \overline{1, T}. \tag{11}$$

Remark that variable y_t may take either sigh. In this model it is only necessary to solve a linear optimization problem.

For the numerical example (see table), the Konno – Yamazaki model has the following form:

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \to \min,$$

under the investment condition (1) and the following 12 inequalities:

$$y_1 + (18.24 - 10.65)x_1 + (12.24 - 11.98)x_2 + (8.23 - 8.34)x_3 + (8.12 - 8.63)x_4 \geq 0,$$
$$y_2 + (12.12 - 10.65)x_1 + (19.16 - 11.98)x_2 + (8.96 - 8.34)x_3 + (8.26 - 8.63)x_4 \geq 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_6 + (10.42 - 10.65)x_1 + (-7.43 - 11.98)x_2 + (8.29 - 8.34)x_3 + (8.95 - 8.63)x_4 \geq 0,$$
$$y_1 - (18.24 - 10.65)x_1 - (12.24 - 11.98)x_2 - (8.23 - 8.34)x_3 - (8.12 - 8.63)x_4 \geq 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_6 - (10.42 - 10.65)x_1 - (-7.43 - 11.98)x_2 - (8.29 - 8.34)x_3 - (8.95 - 8.63)x_4 \geq 0.$$

Solution of this problem with the minimum risk $\sigma(x) = 0.092$ % and return $r(x) = 8.674$ % will be: $x_1 = 0.052\,5$; $x_2 = 0$; $x_3 = 0.212\,5$; $x_4 = 0.735\,0$.

According to Konno and Yamazaki the mean absolute deviation portfolio optimization model's advantages ever the Markowitz's model are (*i*) this model does not use the covariance matrix which therefore does not need to be calculated, (*ii*) solving this linear model is much easier than solving a quadratic model.

In doing so it is possible to differentially penalize the upside from the downside deviation of the portfolio return from its mean. Let p_u and p_d denote penalty parameters for the upside and downside errors respectively. Then constrains (10) and (11) replaced

$$y_t + p_d \sum_{j=1}^{n} (x_{jt} - r_j) x_j \geq 0, \ t = 1, \ldots, T,$$

$$y_t + p_u \sum_{j=1}^{n} (x_{jt} - r_j) x_j \geq 0, \ t = 1, \ldots, T.$$

For the old model is used a symmetric penalty with $p_u = p_d = 1$. It is of particular interest to consider case where $p_u = 0$ and hence the model will penalize only downside risk.

Feinstein and Thapa [7] modified model (9)–(11) proposed a following model that is equivalent to Konno and Yamazaki's:

$$\sum_{t=1}^{T} (u_t + v_t) \to \min$$

subject to (1), (3) and

$$u_t + v_t - \sum_{j=1}^{n} (r_{jt} - r_j) x_j \geq 0, \ t = \overline{1, T},$$

$$u_t, v_t \geq 0, \ t = \overline{1, T}.$$

The calculation by model Feinstein and Thapa gives the result:

$$x_1 = 0.054\,33;\ x_2 = 0;\ x_3 = 0.174\,769;\ x_4 = 0.770\,898.$$

Model based on Minkowski semi-absolute metric of risk estimation

In the standard Markowitz's portfolio model risk is estimated by the standard deviation with Euclidean metric. It is also applicable to use lower semi-variation in order to estimate a portfolio risk:

$$\sigma\_(x) = \sqrt{E\left[R(x) - r(x)\right]^2\_},$$

where $a\_ = \max\{0, -a\}$ losses of expected return are taken into account.

Extension of the semi-variance measure only computed expected return below zero (that is negative returns) or returns below some specific asset such as Tbills, the rate of inflation or a benchmark. These measures of risk implicitly assume that investors want to minimize the damage from returns less than some target risk.

The main point of the model is to find an optimal portfolio in order to minimize sum of out of condition losses [8]. Therefore the risk is estimated by semi-absolute deviation:

$$\sigma(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\min\left\{0, \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j\right\}\right|.$$

Let's assume while choosing the portfolio that if in the history repeats itself then losses will be minimal. The given model is the module of the cautious investor. Certainly it is not applicable if the future tendency is fundamentally different from historical trends.

Let's insert variables y_t each of which represents losses of a portfolio x in the period of time t. Then portfolio optimization problem with semi-absolute deviation can be defined:

$$\sum_{t=1}^{T} y_t \to \min,$$

on conditions that (1), (3) and

$$y_t + \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \geq 0,\ \ y_t \geq 0,\ \ t = \overline{1, T}.$$

Since the model based on a mean semi-absolute deviation risk is bicriterial linear programming model with a smaller number of constraints.

The optimal portfolio in the Konno's model is the following:

$$x_1 = 0.054\ 33, x_2 = 0, x_3 = 0.174\ 77, x_4 = 0.770\ 90$$

with risk $\sigma(x) = 0.088\ 45\ \%$ and return $r(x) = 8.69\ \%$.

Model based on Chebyshev metric of risk estimilation (Maxmin and Minimax model)

Young [9] introduced a minimax portfolio optimization criterion which defines the optimal portfolio as that one that would maximize minimum the return over all the past historical periods. Risk of the portfolio x in this model stands as the measure during the most unsuccessful worst case periods of historical trends, i. e. in metric l_∞:

$$\sigma(x) = \min_{t=1,\dots,T}\sum_{j=1}^{n} r_{jt}x_j \to \max.$$

According to this criterion (4) and budget constraints (1) which is system of linear inequality with parameter λ:

$$\sum_{j=1}^{n} r_{jt}x_j - R_{\min} \geq 0,\ \ t = \overline{1, T},$$

and objective function max λ we get simple bicriterial linear programming problem.

Assuming that $R_{\min} = \sqrt{\frac{2}{\pi}}\sigma_j x_j \to \max$ and replace the risk criterion by a system of linear inequalities with a parameter $R_{\min}$, then the solution of the problem will be: $x_1 = 0$; $x_2 = 0.0382$; $x_3 = 0.0262$; $x_4 = 0.9354$ with risk $\sigma(x) = 0.53\ \%$ and return $r(x) = 8.75\ \%$.

It is worth nothing that Papahristodoulou and Dotzauer [10] compared Markowitz's model and Young's model.

Cai, Teo, Yang and Zhou [11] proposed an alternative minimax risk function in portfolio optimization. The super cautious investor always tries to combine his portfolio proposing that if historical (scenario) situation repeats he should get highest possible earnings from portfolio (losses are minimal in case $R(x)$ is negative value).

Such a risk function is defined as the average of maximum individual risks over number of past time periods, using the maximum absolute deviation risk model l_∞ (Cai's model)

$$\max_{j=1,\dots,n} E\left|R_j x_j - E\left(R_j x_j\right)\right| \to \min.$$

The alternative l_∞ risk function is defined as (Teo's model, see [12]):

$$\frac{1}{4}\sum_{t=1}^{T}\max_{j=1,\ldots,n} E\left|R_{it}x_j - r_{jt}x_j\right| \to \min,$$

$$y \to \min.$$

These models can be transformed into the following linear forms (1), (3) and $y \to \min$

$$E\left|R_j - r_j\right|x_j \le y, \ \ j = 1, \ldots, n,$$

$$\sum_{t=1}^{T} y_t \to \min,$$

$$E\left|R_{jt} - E\left(R_{jt}\right)\right| \le y_t, \ \ t = 1, \ldots, T, \ \ j = 1, \ldots, n.$$

Sharpe model with fractional criteria

The main content of this model is replacement of the bi-criterion model (1), (3), (4) for the one-criterion model with budget constraint (1) and linear-fractional objective function [13]:

$$\frac{\sum_{j=1}^{n} r_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j}} \to \max \ \ \text{(Sharpe-ratio)}.$$

In [14] describes a direct method to obtain the optimal risky portfolio by constructing a convex quadratic programming problem equivalent to Sharpe-ratio. In that form, this problem is not easy to solve. But the "Mathematica" system easily does it by using only one built-in function Maximise. The unique optimal portfolio is ($x_1 = 0.0537$; $x_2 = 0$; $x_3 = 0.1776$; $x_4 = 0.7687$) with risk 0.0883 % and return 8.687 % (the corner portfolio with minimal risk).

Linear models of returns

These models are based on the Sharpe's idea to present expected return function of the market coefficients (market index, GDP, inflation index and etc.). Let it be R_m is the return for the aggregate stock market (market index). More particularly to use single-factor model:

$$r_j = \alpha_j + \beta_j R_m + \varepsilon_j,$$

in which $\beta_j R_m$ assets return r_j is the sum of: linear function with coefficient (beta-coefficient), which shows share sensitivity asset β_j to market trend, constant α_j of the asset j (alpha-coefficient) which doesn't depend on the market conditions and random variable ε_i with $E\left(\varepsilon_i\right) = 0.$ It's supposed that ε_i and R_m are independent, i. e. its covariation is equal zero. In compliance with made assumption expected return of the portfolio x is equal:

$$R\left(x\right) = \sum_{j=1}^{n} x_j\left(\alpha_j + \beta_j E\left(R_m\right) + \varepsilon_j\right)$$

and it's risk

$$\sigma\left(x\right) = \sum_{j=1}^{n} x_j \beta_i^2 \sigma_m^2 + \sum_{j=1}^{n} x_j^2 \sigma_{\varepsilon_j}^2 + \sum_{j\neq 1}\sum_{j\neq 1} x_i x_j \beta_i \beta_j \sigma_m^2.$$

Other more simply criteria firstly assumed by W. Sharpe [2]:

$$\frac{\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n} \beta_j x_j} \to \max.$$

Where β_i is regression coefficient between dividend assets j and market index. Let's give the example of similar function

$$r_j = 0.045 + 0.06\beta_j,\ \beta_j = \frac{\sigma_j \rho_{j\mathrm{DAX}}}{\sigma_{\mathrm{DAX}}},$$

where $\rho_{j\mathrm{DAX}}$ is correlation between asset j and index DAX (30 benchmark German companies) of the historical data.

Model with limited number assets (cardinality constrained)

Generally investors incline to limit number of assets m included in the portfolio. Markowitz' model with additional discrete (boolean) variables δ_j include the following meaning: $\delta_j = 1$ – asset j is putted on the portfolio, $\delta_j = 0$ – asset j is not putted on. Then new constraints are following (a small number of assets):

$$\sum_{j=1}^{n} \delta_j \le m,\ \delta_j \text{ equals 0 or 1},$$

$$x_j \le \delta_j,\ j = \overline{1, n},$$

and new model of portfolio optimization is mixed integer programming problem.

Buy-in thresholds prevent assets from being included in a portfolio with small weights only. They determine that asset weights are either above a lower bound l_j or the asset is not part of the portfolio at all. The main reason for such a constraint might be that some costs are – at least partially-determined by the number of different asset (shares) that are held (e. g. information costs, fixed transaction costs). N. Jobst, M. Horniman, C. Lucas, G. Mitra [15] have shown that a portfolio optimization problem with buy-in thresholds can be formulated as a mixed-integer programming (1)–(4) and supplement constraint:

$$l_j\delta_j \le x_j,\ j = 1, \ldots, n \text{ (thresholds constraint).}$$

For example, it's common for German Investment Law to use constraint (5, 10, 40). The point of this rule is that investor should combine no more than 40 % of mutual funds shares in portfolio, less than 10 % certain type shares in the portfolio and shares of the same issuer are allowed to amount to up to 5 %. These conditions could be modeled by following limits:

$$\sum_{j=1}^{n} x_j \delta_j \le 0.4,\ x_j - 0.05\delta_j \le 0.05,\ j = 1, \ldots, n \text{ (5, 4, 10 – constraint).}$$

Models with transactions costs

In the Markowitz's classical work transaction costs associated with buying and selling of equities were not allowed. The objective is to find the portfolio x that has minimal transactions costs.

Let's bring to the return model transaction costs $d_j x_j$ of the acquiring asset j.

Thus the function of return takes a form:

$$\sum_{j=1}^{n} \left(r_j - d_j\right) x_j \to \max. \tag{12}$$

Inserted variable do not changed an essence of the objective function. Some of the economists give considerations towards the concave function of the transaction costs $d_j\left(x_j\right)$. In this criterion (12) becomes convex.

It is supposed to be more complicated to create a model of fixed costs f_j which do not depend on the size of acquiring assets, f_j is a payment for market entering j. The fixed costs are discrete and it's assumed the inserting of Boolean variables δ_j. The criterion of expected return (3) in this case is replaced on:

$$\sum_{j=1}^{n} \left(r_j x_j - f_j \delta_j\right) \to \max$$

and it adds constraint $x_j \le \delta_j$, δ_j equals 0 or 1, $j = 1, \ldots, n$.

Model with integer (lot) assets

It is supposed under the Markowitz's model that investment capital and its equal 1 and portfolio x combine shares of the assets. At some times shares of the assets could be multiple of the asset value. For instance, at the moment of purchasing asset j has actual price p_j or asset j sells by lots in quantity p_j, $2p_j$, $3p_j$, ….

According to this let insert new variable y_j, which indicate quantity of the asset j, to be included in a portfolio should be an integer multiple of the number of lot, usually 1000 stocks in the Tokyo Stock Exchange. Well the equation (1) should be substituted for inequality:

$$K_0 \le \sum_{j=1}^{n} p_j y_j \le K_1,\ y_j \ge 0 \text{ and integer, } j = \overline{1, n}, \tag{13}$$

where K_0, K_1 are upper and lower limit of the investor's capital. The integer variable y_j represents the number of lots for each asset j which will make part of the optimal portfolio:

$$x_j = \frac{p_j y_j}{\sum_{j=1}^{n} p_j y_j}.$$

Mansini and Speranza [16] present three different heuristics for model (2), (3), (13) with integer variables (using data from the Milan Stock Exchange). The heuristics proposed are based the idea of constructing and solving mixed integer subproblems with consider subsets. The subsets are generated by exploiting the information obtained from the relaxed linear optimization problem.

Integer variables, sometimes also called minimum transaction lots or round lots, are another type of "complex" constrain often mentioned in publications [16–18]. Another way to handle this would be to introduce an asset that represents cash and is divisible up to the smallest currency unit (e. g. cent, ounce of gold). The problem (2), (3), (4) becomes considerably more mathematically compound and concerns to category of the integer quadratic programming.

Model using fuzzy expected return

Choosing optimal portfolios, fuzzy decision theory provides an excellent framework for analysis. Here two reasons: it guaranties a minimum rate of return and gets returns above the risk-free rate for certain market scenarios.

Some authors use fuzzy numbers to represent the future return of assets that approximated as fuzzy numbers the expected return and risk are evaluated by interval-valued means [19; 20]. Let us denote by $\tilde{r}_j$ the fuzzy return on the asset j in the portfolio $P(x)$, then its interval-valued mean is defined as the following interval:

$$E(\tilde{r}_j) = \left[E_*(\tilde{r}_j),\ E^*(\tilde{r}_j)\right].$$

We consider a fuzzy portfolio optimization problem, assuming that the returns assets are modeled by means of a trapezoidal fuzzy number. A fuzzy number $\tilde{A}$ is said to be a trapezoidal fuzzy number $\tilde{A} = (a_u, a_l, c, d)$ if its membership function has the following form (fig. 4).

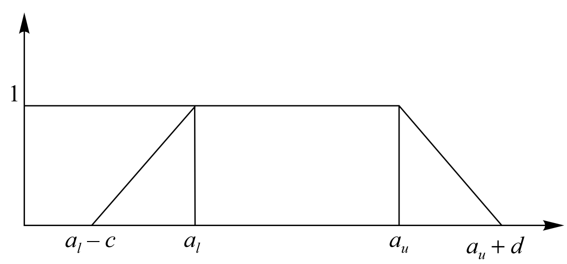

Fig. 4. Trapezoidal fuzzy number

If in addition $a_l = a_u$ it is a triangular fuzzy number.

An essential question connected with solving the fuzzy portfolio optimization problem is related to the defuzzification process for minimization the fuzzy downside for risk considered as a crisp objective and maximize the expected return:

$$\sum_{j=1}^{n}\left(a_{uj} - a_{lj} + \frac{1}{2}(c_j + d_j)\right)x_j \to \min$$

or when the interval-valued possible mean is used, the objective functions are the following:

$$\sum_{j=1}^{n}\left(a_{uj} - a_{lj} + \frac{1}{3}\left(c_j + d_j\right)\right)x_j \to \min,$$

$$\sum_{j=1}^{n}\left(\frac{1}{2}\left(a_{uj} - a_{lj}\right) + \frac{1}{6}\left(d_j - c_j\right)\right)x_j \to \max.$$

Conclusions

The expected return and the risk measured by the variance are the two main characteristics of an optimal portfolio. The optimal portfolio is desirable (the target portfolio). The real portfolio of assets can not be done by human intuition alone and some other characteristics [21]: closeness to the target portfolio; exposure to different economic sectors close to that of the target portfolio; a small number of names; a small number of transactions; high liquidity; low transaction costs.

The mathematical problem can be formulated in many ways but the principal problems can be summarized as follows [22]:

- bicriterial convex quadratic optimization with simple budget constraints;
- bicriterial linear optimization;
- linear optimization with simple polymatroidal budget and risk diversification constraints;
- convex quadratic or linear bicreterial optimization with integer (mixed integer variables).

All models are easily and visually solved by using the "Mathematica" system [23]. That allows to see the optimal variant of capital investments among valid range of solutions.

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