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ПРОИЗВОДСТВЕННЫЕ ФУНКЦИИ С ЗАДАННЫМИ ЭЛАСТИЧНОСТЯМИ ВЫПУСКА И ПРОИЗВОДСТВА

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Рассмотрены обратные задачи восстановления многофакторных производственных функций исходя из заданной эластичности выпуска продукции или эластичности производства. Указаны аналитические виды многофакторных производственных функций с заданной эластичностью выпуска продукции или эластичностью производства. Выделены классы двухфакторных производственных функций, соответствующие заданной (постоянной, линейной, дробно-линейной, степенной и др.) эластичности по капитулу (по труду). Построено множество двухфакторных производственных функций с заданной (постоянной, линейной, дробно-линейной, степенной и др.) эластичностью производства. Полученные результаты могут быть использованы при моделировании реальных производственных процессов.

Ключевые слова: производственная функция; обратная задача; эластичность выпуска; эластичность производства.

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PRODUCTION FUNCTIONS WITH GIVEN ELASTICITIES OF OUTPUT AND PRODUCTION

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In this paper we consider inverse problems of identifying multi-factor production functions from given elasticity of output or from given elasticity of production. The analytical forms of multi-factor production functions with given

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elasticity of output or with given elasticity of production are indicated. Classes of two-factor production functions that correspond to given (constant, linear, linear-fractional, exponential, etc.) elasticity of output with respect to capital (elasticity of output with respect to labour) are obtained. The set of two-factor production functions with given (constant, linear, linear-fractional, exponential, etc.) elasticity of production is built. The obtained results can be applied in modeling of production processes.

Key words: production function; inverse problem; output elasticity; elasticity of production.

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Introduction

Fundamental to economic analysis is the idea of a production function. It and its allied concept, the utility function, form the twin pillars of neoclassical economics [1]. Roughly speaking, the production functions are the mathematical formalization of the relationship between the output of a firm (industry, economy) and the inputs that have been used in obtaining it. In fact, a multi-factor production function is defined as a map

$$Y : x \to f(x) \text{ for all } x = (x_1, \ldots, x_n) \in G, \tag{1}$$

where Y is the quantity of output, n is the number of the inputs, the non-negative function f is a continuously differentiable function on the economic domain G from the first quadrant $\mathrm{R}_+^n = \{(x_1, \ldots, x_n) : x_i > 0,\ i = 1, \ldots, n\}$, and $x_1, x_2, \ldots, x_n$ are the factor inputs (such as labour, capital, land, and raw materials). The production function (1) expresses a technological relationship. It describes the maximum output obtainable, at the existing state of technological knowledge, from given amounts of factor inputs.

At the present time, production functions apply at the level of the individual firm and the macroeconomy at large. At the micro level, economists use production functions to generate cost functions and input demand schedules for the firm. The famous profit-maximizing conditions of optimal factor hire derive from such microeconomics functions. At the level of the macroeconomy, analysts use aggregate production functions to explain the determination of factor income shares and to specify the relative contributions of technological progress and expansion of factor supplies to economic growth [2−5].

The simplest production model used in economics is the famous Cobb – Douglas production function. It was introduced in 1928 by the mathematician Ch. W. Cobb and the economist P. H. Douglas in the paper «A theory of production» [6] in the following form

$$Y : (K, L) \to A K^{\alpha} L^{\beta} \text{ for all } (K, L) \in \mathrm{R}_+^2,\ \mathrm{A} > 0,\ \alpha, \beta \in (0;1),\ \alpha + \beta = 1,$$

where K is the quantity of capital employed, L is the quantity of labour used. We note that in the original definition it is required that $\alpha + \beta = 1$, but this condition has been later relaxed [7]. A generalized Cobb – Douglas production function depending on n-inputs is given by

$$Y : x \to A \prod_{i=1}^{n} x_i^{\alpha_i} \text{ for all } x \in \mathrm{R}_+^n,\ A > 0,\ \alpha_i > 0,\ i = 1, \ldots, n. \tag{2}$$

The Cobb – Douglas production model was generalized in 1961 by the economists K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow [8]. They introduced the so-called constant elasticity of substitution (CES) production function (or the ACMS production function)

$$Y : (K, L) \to (\alpha K^{\gamma} + \beta L^{\gamma})^{\rho/\gamma} \text{ for all } (K, L) \in \mathrm{R}_+^2,\ \alpha, \beta, \rho > 0,\ \gamma \neq 0{,}1.$$

This model was extended to the n-inputs by H. Uzava [9] and D. McFadden [10], who defined a new production function, usually called generalized CES production function

$$Y : x \to A \left(\sum_{i=1}^{n} \alpha_i x_i^{\gamma} \right)^{\rho/\gamma} \text{ for all } x \in \mathrm{R}_+^n,\ A > 0,\ \alpha_i > 0,\ i = 1, \ldots, n,\ \rho > 0,\ \gamma < 1,\ \gamma \neq 0. \tag{3}$$

At the present time, the Cobb – Douglas production function and the CES production function are widely used in economics to represent the relationship of an output to inputs. Specifically, these production functions were used to calculation consumer price index for the Belarusian pharmaceutical market [11] and in hybrid models of economic growth [12] for the Eurasian Economic Union's countries to 2050. Note also that CES production functions include as special cases many other famous production models, like a linear production model, a multinomial production function or Leontief production function.

Concerning the history of development of the theory of production functions see the papers of T. M. Humphrey [1] and S. K. Mishra [13]. For further results concerning new production models in economic see recently articles [14–20]. Many other results on the geometry of production functions can be found in the survey article [21].

For any production function (1), we have a lot of economic-mathematical indicators:

a) the output elasticity with respect to a certain factor of production x_i is defined as

$$\varepsilon_i(x) = \frac{x_i}{f(x)}\,\partial_{x_i} f(x) \equiv \frac{\partial(\ln f(x))}{\partial(\ln x_i)} \text{ for all } x \in G,\ i = 1, \ldots, n;$$

b) the elasticity of production is given by

$$\varepsilon(x) = \lim_{t \to 1} \frac{t}{f(tx)}\,\partial_t f(tx) \equiv \sum_{i=1}^{n} \varepsilon_i(x) \text{ for all } x \in G;$$

c) the marginal rate of technical substitution of input x_j for input x_i is defined by

$$\mathrm{MRS}_{ij} = \frac{\partial_{x_j} f(x)}{\partial_{x_i} f(x)} \text{ for all } x \in G,\ i, j = 1, \ldots, n,\ i \neq j;$$

d) the Hicks elasticity of substitution for the input x_i with respect to the input x_j is

$$H_{ij}(x) = \frac{x_i f_i(x) + x_j f_j(x)}{x_i\, x_j}\ \frac{f_i(x) f_j(x)}{2 f_i(x) f_j(x) f_{ij}(x) - f_i^2(x) f_{jj}(x) - f_j^2(x) f_{ii}(x)} \text{ for all } x \in G,$$

where $f_i = \partial_{x_i} f,\ i = 1, ..., n;\ f_{ij} = \partial^2_{x_i x_j} f,\ i, j = 1, ..., n,\ i \neq j.$

The Cobb – Douglas production function and the CES production function are functions with constant Hicks elasticity of factors substitution. In his paper [14], L. Losonczi proved that a twice differentiable two-factor homogeneous production function with constant Hicks elasticity of substitution is either the Cobb – Douglas production function or the CES production function. This result complements the main propositions of the classical works [7; 8] and is consistent with known results on the classification of production functions [2, p. 111–113]. The analogue for multi-factor production functions was proved by B.-Y. Chen in [15]. These results were recently generalized by G. A. Khatskevich and A. F. Pranevich in [16; 17] for quasi-homogeneous production functions with constant elasticity of factors substitution.

In [20], A. D. Vilcu and G. E. Vilcu classified homogeneous production functions with constant elasticity of labour and capital. Their classification generalized some results [18] by C. A. Ioan and G. Ioan concerning to the sum production function.

The aim of this paper is to identify all multi-factor production functions with given elasticity of output and from given elasticity of production. Also the article is focused on the analytical forms of two-factor production models with given (constant, linear, linear-fractional, exponential, etc.) elasticity of output with respect to capital (elasticity of output with respect to labour) or elasticity of production. For building production functions with specified properties we used the methods of economic analysis and the theory of partial differential equations.

Results and discussion

Multi-factor production function. The following statements describes the analytical form of production functions with given elasticity of output (theorem 1) and from given elasticity of production (theorem 2).

Theorem 1. *Let* $\varepsilon_i : G \to \mathrm{R}$ *be the output elasticity with respect to the factor* x_i, $i \in \{1, \ldots, n\}$, *for some production technology. Then this production technology can be described by one of the production functions of the form*

$$f_\varphi : x \to \varphi\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)\exp\int \frac{\varepsilon_i(x)}{x_i}dx_i \quad for\ all \quad x \in G, \tag{4}$$

where φ *is arbitrary non-negative continuously differentiable function on the domain* R_+^{n-1}.

Proof. Suppose for some production technology we know the output elasticity $\varepsilon_i : G \to R$ with respect to the factor x_i, $i \in \{1, \ldots, n\}$. Then the production function, which is corresponding to this production process, is a solution to the partial differential equation

$$\frac{x_i}{f(x)}\partial_{x_i} f(x) = \varepsilon_i(x).$$

From this first-order partial differential equation, we get

$$\frac{\partial_{x_i} f(x)}{f(x)} = \frac{\varepsilon_i(x)}{x_i},\ \partial_{x_i} \ln f(x) = \frac{\varepsilon_i(x)}{x_i},\ \ln f(x) = \int \frac{\varepsilon_i(x)}{x_i}dx_i + \psi\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right),$$

and

$$f(x) = \exp\left(\int \frac{\varepsilon_i(x)}{x_i}dx_i + \psi\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)\right) = \varphi\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)\exp\int \frac{\varepsilon_i(x)}{x_i}dx_i,$$

where $\varphi = \exp\psi$ and y are arbitrary continuously differentiable functions of the variable *L*.

Therefore the class of production functions with given output elasticity with respect to the factor x_i, $i \in \{1, \ldots, n\}$, has the analytical form (4).

Remark 1. In theorem 1, we assume that the domain $G = \times_{i=1}^{n}(a_i; b_i)$, $0 < a_i < b_i < +\infty$, $i = 1, \ldots, n$, such the function ε_i is continuously differentiable on the closed domain $\bar{G} = \times_{i=1}^{n}[a_i; b_i]$. Under these conditions, we have that the parameter-dependent integral $\int \frac{\varepsilon_i(x)}{x_i}dx_i$ is a continuously differentiable function [22, part II, p. 268–301].

Theorem 2. *Suppose the total elasticity of production* $\varepsilon : G \to R$ *for some production technology is a continuously differentiable function on the domain* $G \subset R_+^n$. *Then this production technology can be described by one of the production functions*

$$f_\varphi : x \to \varphi\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right)\exp\int \frac{\varepsilon(C_1 x_n, \ldots, C_{n-1}x_n, x_n)}{x_n}dx_n\Bigg|_{C_i = \frac{x_i}{x_n},\ i=1,\ldots,n-1} \quad for\ all \quad x \in G, \tag{5}$$

where φ *is arbitrary non-negative continuously differentiable function on the domain* R_+^{n-1}.

Proof. Suppose for some production technology we know the total elasticity of production $\varepsilon : G \to R$. Then the production function, which is corresponding to this production process, is a solution to the first-order partial differential equation

$$\sum_{i=1}^{n} x_i \partial_{x_i} f(x) = f(x)\varepsilon(x). \tag{6}$$

The ordinary differential system in the symmetric form

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \ldots = \frac{dx_n}{x_n} = \frac{df}{\varepsilon(x)f} \tag{7}$$

corresponds to the quasilinear partial differential equation of the first order (5).

Integrating the differential equations

$$\frac{dx_i}{x_i} = \frac{dx_n}{x_n},\ i = 1, \ldots, n-1,$$

we obtain $\frac{x_i}{x_n} = C_i$, where C_i is arbitrary real constant, $i = 1, \ldots, n-1$.

Therefore, the rational functions

$$\xi_i : x \to \frac{x_i}{x_n} \text{ for all } x \in \mathrm{R}_+^{n-1},\ i = 1, \ldots, n-1,$$

are first integrals of the differential system (7).

Since $x_i = C_i x_n,\ i = 1, \ldots, n-1,$ we see that from the differential equation

$$\frac{dx_n}{x_n} = \frac{df}{\varepsilon(x) f}$$

it follows that

$$\frac{df}{f} = \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right) dx_n}{x_n},\quad \ln f = \int \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right) dx_n}{x_n} + \tilde{C}_n,$$

$$f = C_n \exp \int \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right) dx_n}{x_n},\quad f \exp\left(-\int \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right) dx_n}{x_n}\right) = C_n,$$

where $\tilde{C}_n$ and $C_n = \exp \tilde{C}_n$ are arbitrary real constants. Therefore, the function

$$\xi_n : x \to f \exp\left(-\int \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right) dx_n}{x_n}\right) \text{ for all } x \in G$$

is a first integral of the differential system in the symmetric form (7).

Using the functionally independent first integrals of system (7), we can build the general solution to the first-order partial differential equation (6)

$$\Phi\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n},\ f \exp\left(-\int \frac{\varepsilon\left(C_1 x_n, \ldots, C_{n-1} x_n, x_n\right)}{x_n} dx_n\right)\Bigg|_{C_i = \frac{x_i}{x_n},\ i = 1, \ldots, n-1}\right) = 0, \qquad (8)$$

where Φ is arbitrary continuously differentiable function. If we take functions Φ such that the equation (8) is solvable with respect to the last argument (see the implicit function theorem, for example, in [22, part I, p. 544–551]), then from the equation (8), we get the solution to the equation (6) in explicit form (5).

For example, if the total elasticity of production $\varepsilon : x \to \alpha_0$ for all $x \in G,\ \alpha_0 \in \mathrm{R},$ then this production technology can be described by one of the production functions of the form

$$f_\varphi : x \to x_n^{\alpha_0}\, \varphi\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right) \text{ for all } x \in G.$$

Using this formula, we obtain the following statements:

a) if $\alpha_0 = 1$ and $\varphi : \xi \to \sum_{i=1}^{n-1} \alpha_i \xi_i + \alpha_n$ for all $\xi \in \mathrm{R}_+^{n-1},$ then we have the linear function

$$f : x \to \sum_{i=1}^{n} \alpha_i x_i \text{ for all } x \in \mathrm{R}_+^n,\ \alpha_i \in \mathrm{R}_+,\ i = 1, \ldots, n;$$

b) if $\varphi : x \to A \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_{n-1}^{\alpha_{n-1}}$ for all $\xi \in \mathrm{R}_+^{n-1},$ then we get the Cobb – Douglas production function (2) with $\alpha_n = \alpha_0 - \sum_{i=1}^{n-1} \alpha_i;$

c) if $\varphi : x \to A\left(\sum_{i=1}^{n-1} \alpha_i \xi^{\gamma} + \alpha_n\right)^{\alpha_0/\gamma}$ for all $\xi \in \mathrm{R}_+^{n-1},$ then we obviously have the CES production function (3) with $\rho = \alpha_0.$

Two-factor production function. Now let us consider a two-factor production function

$$Y:(K,\,L)\to f(K,\,L)\ \text{ for all }\ (K,\,L)\in G,\,G\subset \mathrm{R}_{+}^{2},$$

where K is the quantity of capital employed, L is the quantity of labour used, Y is the quantity of output, and the non-negative function f is a continuously differentiable function on G.

Taking into account theorems 1 and 2 for the multi-factor case, we obtain the following assertions for two-factor production functions (theorems 3–5).

Theorem 3. *Suppose* $\varepsilon_K:(K,\,L)\to\varepsilon_K(K,\,L)$ *for all* $(K,\,L)\in G$ *is the output elasticity with respect to the capital K for some production technology. Then this production technology can be described by one of the production functions of the form*

$$f_{\varphi}:(K,\,L)\to\varphi(L)\exp\int\frac{\varepsilon_K(K,\,L)}{K}dK\ \ \textit{for all}\ \ (K,\,L)\in G,$$

where φ *is arbitrary non-negative continuously differentiable function on the interval* $(0;+\infty)$.

Using theorem 3, from some given output elasticities with respect to the capital, we obtain the corresponding classes of production functions (see table 1).

Table 1

The form of production function with given elasticity of output with respect to capital

No.	Elasticity of output with respect to capital $(\alpha,\beta,\gamma\in\mathrm{R},u,v\in C(G))$	Analytical form of production function
1	$\varepsilon_K(K,L)=\gamma$	$f_{\varphi}(K,L)=\varphi(L)K^{\gamma}$
2	$\varepsilon_K(K,L)=\alpha K+\beta L+\gamma$	$f_{\varphi}(K,L)=\phi(L)K^{\beta L+\gamma}e^{\alpha K}$
3	$\varepsilon_K(K,L)=u(K)+v(L)$	$f_{\varphi}(K,L)=\varphi(L)K^{v(L)}\exp\int\frac{u(K)}{K}dK$
4	$\varepsilon_K(K,L)=K^{\alpha}u\left(\frac{K}{L}\right)$	$f_{\varphi}(K,L)=\varphi(L)\exp\left(L^{\alpha}\int\xi^{\alpha-1}u(\xi)d\xi\right)_{\mid\,\xi=\frac{K}{L}}$
5	$\varepsilon_K(K,L)=L^{\beta}u\left(\frac{K}{L}\right)$	$f_{\varphi}(K,L)=\varphi(L)\exp\left(L^{\beta}\int\frac{u(\xi)}{\xi}d\xi\right)_{\mid\,\xi=\frac{K}{L}}$
6	$\varepsilon_K(K,L)=u\left(K^{\alpha}L^{\beta}\right),\quad\alpha,\beta\neq 0$	$f_{\varphi}(K,L)=\varphi(L)\exp\left(\frac{1}{\alpha}\int\frac{u(\xi)}{\xi}d\xi\right)_{\mid\,\xi=K^{\alpha}L^{\beta}}$

Note. Developed by the authors.

Theorem 4. *Suppose* $\varepsilon_L:(K,L)\to\varepsilon_L(K,L)$ *for all* $(K,L)\in G$ *is the output elasticity with respect to the labour L for some production technology. Then this production technology can be described by one of the production functions of the form*

$$f_{\varphi}:(K,L)\to\varphi(K)\exp\int\frac{\varepsilon_L(K,L)}{L}dL\ \ \textit{for all}\ \ (K,L)\in G,$$

where φ *is arbitrary non-negative continuously differentiable function on the interval* $(0;+\infty)$.

Using theorem 4, from some given output elasticities with respect to the labour, we obtain the corresponding classes of production functions (see table 2).

Table 2

The form of production function with given elasticity of output with respect to labour

No.	Elasticity of output with respect to labour $(\alpha, \beta, \gamma \in \mathrm{R},\ u, v \in C(G))$	Analytical form of production function
1	$\varepsilon_L(K,L) = \gamma$	$f_\varphi(K,L) = \varphi(K)L^\gamma$
2	$\varepsilon_L(K,L) = \alpha K + \beta L + \gamma$	$f_\varphi(K,L) = \varphi(K)L^{\alpha K + \gamma}e^{\beta L}$
3	$\varepsilon_L(K,L) = u(K) + v(L)$	$f_\varphi(K,L) = \varphi(K)L^{u(K)}\exp\int \frac{v(L)}{L}dL$
4	$\varepsilon_L(K,L) = K^\alpha u\left(\frac{K}{L}\right)$	$f_\varphi(K,L) = \varphi(K)\exp\left(-K^\alpha \int \frac{u(\xi)}{\xi}d\xi\right)_{\mid \xi = \frac{K}{L}}$
5	$\varepsilon_L(K,L) = L^\beta u\left(\frac{K}{L}\right)$	$f_\varphi(K,L) = \varphi(K)\exp\left(-K^\beta \int \frac{u(\xi)}{\xi^{\beta+1}}d\xi\right)_{\mid \xi = \frac{K}{L}}$
6	$\varepsilon_L(K,L) = u\left(K^\alpha L^\beta\right), \quad \alpha, \beta \neq 0$	$f_\varphi(K,L) = \varphi(K)\exp\left(\frac{1}{\beta}\int \frac{u(\xi)}{\xi}d\xi\right)_{\mid \xi = K^\alpha L^\beta}$

Note. Developed by the authors.

Theorem 5. *Suppose* $\varepsilon : (K,L) \to \varepsilon(K,L)$ *for all* $(K,L) \in G$ *is the total elasticity of production for some production technology. Then this production technology can be described by one of the production functions of the form*

$$f_\varphi : (K,L) \to \varphi\left(\frac{K}{L}\right)\exp\left(\int \frac{\varepsilon(C_1 L, L)}{L}dL\right)_{\mid C_1 = \frac{K}{L}} \text{ for all } (K,L) \in G,$$

where φ *is arbitrary non-negative continuously differentiable function on the interval* $(0;+\infty)$.

Using theorem 5, from some given total elasticities of production, we obtain the corresponding classes of production functions (see table 3).

Table 3

The form of production function with given total elasticity of production

No.	Total elasticity of production $(\alpha, \beta, \gamma \in \mathrm{R},\ u, v \in C(G))$	Analytical form of production function
1	$\varepsilon(K,L) = \gamma$	$f_\varphi(K,L) = L^\gamma \varphi\left(\frac{K}{L}\right)$
2	$\varepsilon(K,L) = \alpha K + \beta L + \gamma$	$f_\varphi(K,L) = L^\gamma \varphi\left(\frac{K}{L}\right)\exp(\alpha K + \beta L)$
3	$\varepsilon(K,L) = u(\alpha K + \beta L)$	$f_\varphi(K,L) = \varphi\left(\frac{K}{L}\right)\exp\left(\int \frac{u(\xi)}{\xi}d\xi\right)_{\mid \xi = \alpha K + \beta L}$
4	$\varepsilon(K,L) = u(K) + v(L)$	$f_\varphi(K,L) = \varphi\left(\frac{K}{L}\right)\exp\left(\int \frac{u(K)}{K}dK + \int \frac{v(L)}{L}dL\right)$

Ending table 3

No.	Total elasticity of production $(\alpha, \beta, \gamma \in \mathrm{R}, \quad u, v \in C(G))$	Analytical form of production function
5	$\varepsilon(K, L) = u\left(\frac{K}{L}\right)$	$f_\varphi(K, L) = \varphi\left(\frac{K}{L}\right)\exp\left(\ln L \cdot u\left(\frac{K}{L}\right)\right)$
6	$\varepsilon(K, L) = K^\alpha u\left(\frac{K}{L}\right), \ \alpha \neq 0$	$f_\varphi(K, L) = \varphi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\alpha} K^\alpha u\left(\frac{K}{L}\right)\right)$
7	$\varepsilon(K, L) = L^\beta f\left(\frac{K}{L}\right), \beta \neq 0$	$f_\varphi(K, L) = \varphi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\beta} L^\beta u\left(\frac{K}{L}\right)\right)$
8	$\varepsilon(K, L) = u\left(K^\alpha L^\beta\right), \ \alpha \neq -\beta$	$f_\varphi(K, L) = \varphi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\alpha + \beta}\int \frac{u(\xi)}{\xi} d\xi\right)_{\mid \xi = K^\alpha L^\beta}$

Note. Developed by the authors.

Conclusions

In this article we completely classify multi-factor production functions with given elasticity of output (theorem 1) and multi-factor production functions with given elasticity of production (theorem 2). Classes of two-factor production functions that correspond to given (constant, linear, linear-fractional, exponential, etc.) elasticity of output with respect to capital (theorem 3 and table 1) and labour (theorem 4 and table 2) are obtained. Full set of two-factor production functions with given (constant, linear, linear-fractional, exponential, etc.) elasticity of production (theorem 5 and table 3) is built. The obtained theoretical results may be useful in economic modeling of production at the regional and country levels.

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