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О НЕКОТОРЫХ КЛАССАХ ПОДРЕШЕТОК РЕШЕТКИ ВСЕХ ПОДГРУПП

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В настоящей статье G всегда обозначает группу. Если K и H – подгруппы группы G , где K – нормальная подгруппа группы H , то фактор-группа группы H по K называется секцией группы G . Такая секция является нормальной, если K и H – нормальные подгруппы группы G , и тривиальной, если K и H равны. Назовем произвольное множество Σ нормальных секций группы G расслоением группы G , если оно содержит каждую тривиальную нормальную секцию группы G , и будем говорить, что расслоение Σ группы G является G -замкнутым, если Σ содержит каждую такую нормальную секцию группы G , которая G -изоморфна некоторой нормальной секции группы G , принадлежащей множеству Σ . Пусть теперь Σ – произвольное G -замкнутое расслоение группы G и пусть \mathcal{L} – множество всех таких подгрупп A группы G , что фактор-группа группы V по W , где V – нормальное замыкание A в G , а W – нормальное ядро A в G , принадлежит Σ . Опишем условия на Σ , при которых множество \mathcal{L} является подрешеткой решетки всех подгрупп группы G , а также обсудим некоторые применения этой подрешетки в теории обобщенных конечных T -групп.

Ключевые слова: группа; решетка подгрупп; модулярная решетка; формационное множество Фиттинга; формация Фиттинга.

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ON SOME CLASSES OF SUBLATTICES OF THE SUBGROUP LATTICE

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In this paper G always denotes a group. If K and H are subgroups of G , where K is a normal subgroup of H , then the factor group of H by K is called a section of G . Such a section is called normal, if K and H are normal subgroups of G , and trivial, if K and H are equal. We call any set Σ of normal sections of G a stratification of G , if Σ contains every trivial normal section of G , and we say that a stratification Σ of G is G -closed, if Σ contains every such a normal section of G , which is G -isomorphic to some normal section of G belonging Σ . Now let Σ be any G -closed stratification of G , and let \mathcal{L} be the set of all subgroups A of G such that the factor group of V by W , where V is the normal closure of A in G and W is the normal core of A in G , belongs to Σ . In this paper we describe the conditions on Σ under which the set \mathcal{L} is a sublattice of the lattice of all subgroups of G and we also discuss some applications of this sublattice in the theory of generalized finite T -groups.

Keywords: group; subgroup lattice; modular lattice; formation Fitting set; Fitting formation.

Introduction

In this paper G always denotes a group. Moreover, $\mathcal{L}(G)$ denotes the set (the lattice) of all subgroups of G and $\mathcal{L}_n(G)$ is the set (the lattice) of all normal subgroups of G . In this paper \mathfrak{F} is a class of groups containing all identity groups, \mathfrak{N}^* is the class of all finite quasinilpotent groups, \mathfrak{N} is the class of all finite nilpotent groups and \mathfrak{U} is the class of all finite supersoluble groups.

A class of groups \mathfrak{F} is said to be a *Fitting formation* if the following conditions hold: (1) for every normal subgroup N of any group $G \in \mathfrak{F}$ both groups N and G/N belong to \mathfrak{F} ; (2) $G \in \mathfrak{F}$ whenever G has normal subgroups A and B and either $G/A, G/B \in \mathfrak{F}$ and $A \cap B = 1$ or $G = AB$ and $A, B \in \mathfrak{F}$.

One of the organizing ideas of the group theory is the idea to study the group G depending on the presence in it a subgroup system \mathcal{L} having desired properties. Such an approach is the most effective in the case when \mathcal{L} forms a *sublattice* of $\mathcal{L}(G)$, that is, $A \cap B \in \mathcal{L}$ and $\langle A, B \rangle \in \mathcal{L}$ for all $A, B \in \mathcal{L}$. This circumstance makes the general problem of finding sublattices in $\mathcal{L}(G)$ important and interesting.

One of the first results in this direction was obtained by Wielandt in his paper [1], where it was proved that the set $\mathcal{L}_{sn}(G)$ of all subnormal subgroups of the group G having a composition series is a sublattice of $\mathcal{L}(G)$. In the case when G is finite, an original generalization of the lattice $\mathcal{L}_{sn}(G)$ was found by Kegel [2]. A subgroup A of G is called \mathfrak{F} -subnormal in G in the sense of Kegel [2] or K - \mathfrak{F} -subnormal in G [3, definition 6.1.4], if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i} \in \mathfrak{F}$ for all $i = 1, \dots, t$. Kegel proved [2] that if the class \mathfrak{F} is closed under extensions, epimorphic images and subgroups, then the set $\mathcal{L}_{\mathfrak{F}sn}(G)$ of all K - \mathfrak{F} -subnormal subgroups of a finite group G is a sublattice of the lattice $\mathcal{L}(G)$. For every set π of primes, we may choose the class \mathfrak{F} of all π -groups. In this way we obtain infinitely many functors $\mathcal{L}_{\mathfrak{F}sn}$ assigning to every finite group G a sublattice of $\mathcal{L}(G)$ containing $\mathcal{L}_{sn}(G)$. Subsequently, this result was generalized (also in the universe of all finite groups) on the basis of methods of the formation theory (see, in particular, [4; 5] and chapter 6 in [3]).

In this paper, we develop a new approach for finding sublattices in $\mathcal{L}(G)$, where G is an arbitrary group, and we also discuss some applications of such sublattices.

The main concepts and results

If $K \trianglelefteq H \leq G$, then H/K is called a *section* of G ; such a section is called: *normal* if H and K are normal subgroups of G ; *trivial* if $H = K$; a *chief factor* of G provided $K < H$ and for any normal subgroup L of G with $K \leq L \leq H$ we have either $K = L$ or $L = H$. We write $H/K \simeq_G T/L$ provided the normal sections H/K and T/L of G are G -isomorphic; $Ch_G(H/K)$ denotes the set of all chief factors T/L of G with $K \leq L < T \leq H$; A^G is the normal closure of the subgroup A in G and $A_G = \bigcap_{x \in G} A^x$. If Δ is any set of chief factors of G (not necessary non-empty),

then we write $\Sigma_G(\Delta)$ to denote the set of all normal sections H/K of G such that either $K = H$ or $K < H$ and the series $K < H$ can be refined to a chief series of G between K and H (of finite length) with $Ch_G(H/K) \subseteq \Delta$.

We call a set Σ of normal sections of G a *stratification* of G if Σ contains every trivial normal section of G and we say that a stratification Σ of G is *G-closed* provided $H/K \in \Sigma$ whenever H/K is a normal section of G with $H/K \simeq_G T/L \in \Sigma$.

Now let Σ be any stratification of G . Then write $\mathfrak{L}_\Sigma(G)$ to denote the set of all subgroups A of G with $A^G/A_G \in \Sigma$.

We will use $\Sigma_G(\mathfrak{F})$ to denote the set of normal sections H/K of G such that $H/K \in \mathfrak{F}$.

Definition. We say (by analogy with the definition of the *Fitting set* of a group [6, p. 537]) that a G -closed stratification Σ of G is a *formation Fitting set* of G if the following conditions hold:

- (i) for every two normal sections H/K and T/K of G where $T/K \in \Sigma$ and $H \leq T$, we have $H/K, T/H \in \Sigma$;
- (ii) $H/(K \cap N) \in \Sigma$ for every two sections $H/K, H/N \in \Sigma$;
- (iii) $HV/K \in \Sigma$ for every two sections $H/K, V/K \in \Sigma$.

The usefulness of this concept is primarily based on the following our three results.

Theorem 1. *If $\Sigma = \Sigma_G(\Delta)$ for some G -closed set Δ of chief factors of G or $\Sigma = \Sigma_G(\mathfrak{F})$ for some Fitting formation \mathfrak{F} , then Σ is a formation Fitting set of G .*

Theorem 2. *The set $\mathfrak{L}_\Sigma(G)$ forms a sublattice in $\mathfrak{L}(G)$ for each formation Fitting set Σ of G .*

Theorem 3. *The inclusion $\mathfrak{L}_n(G) \subseteq \mathfrak{L}_\Sigma(G)$ holds for every formation Fitting set Σ of G . Moreover, in the case when G satisfies the maximality condition the lattice $\mathfrak{L}_\Sigma(G)$ is distributive if and only if $\mathfrak{L}_\Sigma(G) = \mathfrak{L}_n(G)$ is distributive.*

From theorems 1 and 2 we get the following.

Corollary 1. *Let \mathfrak{F} be either the class of all nilpotent groups, or the class of all soluble groups, or the class of all finite quasinilpotent groups. Then the set $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ forms a sublattice in $\mathfrak{L}(G)$.*

We say that a chief factor H/K of G is *\mathfrak{F} -central* in G [7] if

$$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}.$$

Let $D = M \rtimes A$ and $R = N \rtimes B$. Then the pairs (M, A) and (R, B) are said to be *equivalent* provided there are isomorphisms $f: M \rightarrow N$ and $g: A \rightarrow B$ such that $f(a^{-1}ma) = g(a^{-1})f(m)g(a)$ for all $m \in M$ and $a \in A$.

In fact, the following lemma is known (see, for example, lemma 3.27 in [7]) and it can be proved by the direct verification.

Lemma 1. *Let $D = M \rtimes A$ and $R = N \rtimes B$. If the pairs (M, A) and (R, B) are equivalent, then $D = R$.*

Lemma 2. *Let N, M and $K < H \leq G$ be normal subgroups of G , where H/K is a chief factor of G :*

- (1) *if $N \leq K$, then $(H/K) \rtimes (G/C_G(H/K)) \simeq ((H/N)/(K/N)) \rtimes ((G/N)/C_{G/N}((H/N)/(K/N)))$;*
- (2) *if T/L is a chief factor of G and H/K and T/L are G -isomorphic, then $C_G(H/K) = C_G(T/L)$ and $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$;*
- (3) *$(MN/N) \rtimes (G/C_G(MN/N)) \simeq (M/M \cap N) \rtimes (G/C_G(M/M \cap N))$.*

Proof. (1) In view of the G -isomorphisms $H/K \simeq (H/N)/(K/N)$ and

$$G/C_G(H/K) \simeq (G/N)/(C_G(H/K)/N),$$

the pairs

$$(H/K, G/C_G(H/K)), ((H/N)/(K/N), (G/N)/C_{G/N}((H/N)/(K/N)))$$

are equivalent. Hence statement (1) is a corollary of lemma 1.

(2) A direct check shows that $C = C_{G/N}(H/K) = C_G(T/L)$ and that the pairs $(H/K, G/C)$ and $(T/L, G/C)$ are equivalent. Hence statement (2) is also a corollary of lemma 1.

(3) This follows from the G -isomorphism $MN/N \simeq M/M \cap N$ and part (2).

The lemma is proved.

In view of lemma 2, we get from theorems 1 and 2 the following fact.

Corollary 2. *Let Δ be the set of all \mathfrak{F} -central chief factors of G . Then the set $\mathfrak{L}_{\Sigma(\Delta)}(G)$ forms a sublattice in $\mathfrak{L}(G)$.*

Remark 1. (i) Let $\Sigma(G)$ be the set of all formation Fitting sets of G . It is clear that $\Sigma(G)$ is partially ordered with respect to set inclusion and the formation Fitting set $\{H/K \mid H, K \in \mathfrak{L}_n(G)\}$ is the greatest element in $\Sigma(G)$. Moreover, for every set $\{\Sigma_i \mid i \in I\}$ of formation Fitting sets of G the intersection $\bigcap_{i \in I} \Sigma_i$ is also a formation Fitting set of G and so $\bigcap_{i \in I} \Sigma_i$ is the greatest lower bound for $\{\Sigma_i \mid i \in I\}$ in $\Sigma(G)$. Therefore $\Sigma(G)$ is a complete lattice. The set $\{H/H \mid H \trianglelefteq G\}$ is the smallest element in $\Sigma(G)$.

(ii) Let \mathfrak{X} be any set of normal sections of G . Then the set $\{\Sigma_i \mid i \in I\}$ of all formation Fitting sets of G containing \mathfrak{X} is non-empty and the intersection $\bigcap_{i \in I} \Sigma_i$ is a formation Fitting set of G by part (i). We say that $\bigcap_{i \in I} \Sigma_i$ is the *formation Fitting set of G generated by \mathfrak{X}* and denote it by $\text{formfit}(\mathfrak{X})$. If $\mathfrak{X} = \{T/L\}$ is a singleton set, we write $\text{formfit}(T/L)$ instead of $\text{formfit}(\{T/L\})$ and say that $\text{formfit}(T/L)$ is a *one-generated formation Fitting set of G* .

(iii) Let E and N be subgroups of G , where N is normal in G . Then for any stratification Σ of G we use $\Sigma N/N$ and $\Sigma \cap E$ to denote the stratification $\{(NH/N)/(NK/N) \mid H/K \in \Sigma\}$ of G/N and the stratification $\{(T \cap E)/(L \cap E) \mid T/L \in \Sigma\}$ of E , respectively. If Σ is a formation Fitting set of G , then $\Sigma N/N$ is a formation Fitting set of G/N (see proposition (iv) below).

From theorem 1 we get the following useful result.

Corollary 3. *Let \mathfrak{X} be a set of normal sections of G and $T/L \in \Sigma = \text{formfit}(\mathfrak{X})$. Then the following statements hold:*

(i) $T/L \in \mathfrak{F}$ for every Fitting formation \mathfrak{F} containing \mathfrak{X} ;

(ii) if $H/K \in \text{Ch}(T/L)$, then $H/K \cong_G F/S$ for some $F/S \in \text{Ch}(V/W)$ and $V/W \in \mathfrak{X}$.

For any two sections H/K and T/L of G we write $H/K \leq T/L$ provided $K \leq L$ and $H \leq T$. Then the set of all sections of G is partially ordered with respect to \leq .

The proofs of theorems 2 and 3 are based on the following useful observation.

Proposition. *Let Σ be a formation Fitting set of G and let E and N be subgroups of G , where $N \trianglelefteq G$. Then:*

(i) $\langle \Sigma, \leq \rangle$ is a lattice in which HV/KW is the least upper bound and $(H \cap V)/(K \cap W)$ is the greatest lower bound of $\{H/K, V/W\}$ for any two its sections $H/K, V/W$;

(ii) if $T/L \in \Sigma$, then $\mathfrak{L}(T/L)$ is isomorphic to the interval $[T, L]$ in $\mathfrak{L}_\Sigma(G)$;

(iii) if $f: G \rightarrow G^*$ is an isomorphism, then $f(\Sigma) := \{T^f/L^f \mid T/L \in \Sigma\}$ is a formation Fitting set of G^* . Moreover, if Σ is hereditary, then $f(\Sigma)$ is hereditary;

(iv) $\Sigma N/N$ is a formation Fitting set of G/N and $\Sigma N/N = \{(H/N)/(K/N) \mid H/K \in \Sigma \text{ and } N \leq K\}$.

Proof. (i) Since $H/K \in \Sigma$ and $K(V \cap H)/K \leq H/K$, we have $K(V \cap H)/K \in \Sigma$. Hence from the G -isomorphism

$$(H \cap V)/(K \cap V) = (H \cap V)/(K \cap V \cap H) \cong K(V \cap H)/K$$

we get that $(H \cap V)/(K \cap V) \in \Sigma$. Similarly, $(V \cap H)/(W \cap H) \in \Sigma$. But then we get that

$$(H \cap V)/((K \cap V) \cap (W \cap H)) = (H \cap V)/(K \cap W) \in \Sigma$$

since Σ is a formation Fitting set of G by hypothesis.

From the G -isomorphism

$$H(KW)/KW \cong H/(H \cap KW) = H/K(H \cap W)$$

we get that $HKW/KW \in \Sigma$ since $(H \cap W)K/K \leq H/K$. Similarly, one can get that $VKW/KW \in \Sigma$. Moreover,

$$HV/KW = (HKW/KW)(VKW/KW)$$

and so $HV/KW \in \Sigma$. Hence statement (i) holds.

(ii) This statement follows from the fact that for every subgroup H of G with $L \leq H \leq T$ we have $L \leq H_G$ and $H^G \leq T$.

(iii) This assertion can be proved by direct checking.

(iv) First note that, in view of part (i), $V/W \in \Sigma$ always implies that $VN/WN \in \Sigma$, so every normal section of G/N in $\Sigma N/N$ is of the form $(V/N)/(W/N)$ for some $V/W \in \Sigma$.

(1) $\Sigma N/N$ is (G/N) -closed.

Indeed, if

$$(H/N)/(K/N) \approx_{G/N} (V/N)/(W/N) \in \Sigma N/N,$$

then $H/K \approx_G (V/W) \in \Sigma$. Hence $H/K \in \Sigma$, so $(H/N)/(K/N) \in \Sigma N/N$.

(2) For every two normal sections $(H/N)/(K/N)$ and $(T/N)/(K/N)$ of G/N , where $H/N \leq T/N$ and $(T/N)/(K/N) \in \Sigma N/N$ both sections $(H/N)/(K/N)$ and $(T/N)/(H/N)$ belong to $\Sigma N/N$. (This assertion is evident.)

(3) $(H/N)/((K/N) \cap (L/N)) \in \Sigma N/N$ for every two normal sections $(H/N)/(K/N), (H/N)/(L/N) \in \Sigma N/N$.

From

$$(H/N)/(K/N), (H/N)/(L/N) \in \Sigma N/N$$

we get that $H/K, H/L \in \Sigma$ and so $H/(K \cap L) \in \Sigma$, which implies that

$$(H/N)/((K/N) \cap (L/N)) = (H/N)/((K \cap L)/N) \in \Sigma N/N.$$

(4) $(H/N)(V/N)/(K/N) \in \Sigma N/N$ for every two normal sections $(H/N)/(K/N), (V/N)/(K/N) \in \Sigma N/N$.

From $(H/N)/(K/N), (V/N)/(K/N) \in \Sigma N/N$ it follows that $HV/K \in \Sigma$, which implies that $(H/N)(V/N)/(K/N) \in \Sigma N/N$.

Hence statement (iv) holds.

The proposition is proved.

Before proceeding, consider some examples.

Example 1. (i) If $\mathfrak{X} = \{G/1\}$, then

$$\text{formfit}(G/1) = \{H/K \mid H, K \leq G\}$$

and so

$$\mathfrak{L}_{\text{formfit}(G/1)}(G) = \mathfrak{L}(G).$$

(ii) If \mathfrak{F} is the class of all identity groups, then $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G) = \mathfrak{L}_n(G)$.

(iii) Let $p > q > 2$ be primes, where q divides $p - 1$. Let Q be a non-abelian group of order q^3 . Then Q has a unique minimal normal subgroup, so there exists a simple $\mathbb{F}_p Q$ -module P which is faithful for Q . Then $|P| > p$. Let $G = (P \rtimes Q) \times (C_p \rtimes C_q)$, where $C_p \rtimes C_q$ is a non-abelian group of order pq . Let Δ is the set of all those chief factors of G on which G induces an abelian group of automorphisms. Then

$$\mathfrak{L}(P) \not\subseteq \mathfrak{L}_{\Sigma_G(\Delta)}(G) = \mathfrak{L}_n(G) \cup \{AC_q^x \mid A \trianglelefteq G, x \in G\}.$$

Therefore for every Fitting formation \mathfrak{F} we have $\mathfrak{L}_{\Sigma_G(\Delta)}(G) \neq \mathfrak{L}_{\Sigma_G(\mathfrak{F})}$ since otherwise $P \in \mathfrak{F}$ and so

$$\mathfrak{L}(P) \subseteq \mathfrak{L}_{\Sigma_G(\mathfrak{F})} = \mathfrak{L}_{\Sigma_G(\Delta)}(G).$$

(iv) Let A be a non-abelian simple group and \mathfrak{F} the class of all groups B such that either $B = 1$ or B is the direct product of isomorphic copies of A . Let $G = A_0 \wr A = K \rtimes A$, where $A_0 \simeq A$ and $K = A_1 \times \cdots \times A_{|A|}$ is the base group of the regular wreath product G . Then K is the unique minimal normal subgroup of G by [6, chapter A, proposition 18.5]. Moreover,

$$\Sigma := \Sigma_G(\mathfrak{F}) = \{G/K, K/1, G/G, K/K, 1/1\}$$

is clearly a formation Fitting set of G , so $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ is a sublattice of $\mathfrak{L}(G)$. We show that $\mathfrak{L}_{\Sigma_G(\mathfrak{F})} \neq \mathfrak{L}_{\Sigma_G(\Delta)}(G)$ for every G -closed set Δ of chief factors of G . Indeed, assume that $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G) = \mathfrak{L}_{\Sigma_G(\Delta)}(G)$. Then for all subgroups $L \leq K$ and $K \leq R \leq G$ we have $L^G/L_G = K/1$ and $R^G/R_G = G/K$, so $L, R \in \mathfrak{L}_{\Sigma_G(\Delta)}(G)$. Therefore $R/1, G/K \in \Delta$ and

hence $G/1 \in \Sigma_G(\Delta)$. Thus $\mathfrak{L}_{\Sigma_G(\Delta)}(G) = \mathfrak{L}(G)$ and so $A \in \mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$. But then $G/1 = A^G/A_G \in \mathfrak{F}$, which means that G is the direct product of isomorphic copies of A . This contradiction shows that

$$\mathfrak{L}_{\Sigma_G(\mathfrak{F})} \neq \mathfrak{L}_{\Sigma_G(\Delta)}(G)$$

for every G -closed set Δ of chief factors of G .

(v) The class of groups \mathfrak{F} is called a *saturated* if \mathfrak{F} contains every finite group G with $G/\Phi(G) \in \mathfrak{F}$.

Now let A be a maximal subgroup of a finite group G and let \mathfrak{F} be a saturated Fitting formation. Let Δ be the set of all \mathfrak{F} -central chief factors of G . Then $G/A_G = A^G/A_G \in \mathfrak{F}$ if and only if $A^G/A_G \in \Sigma_G(\Delta)$ (see lemma 5 below). Therefore $A \in \mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ if and only if $A \in \mathfrak{L}_{\Sigma_G(\Delta)}(G)$.

In conclusion of this section note that some special versions of theorems 2 and 3 were proved in the papers [8–10]. In particular, in the paper [9], the following results were proved.

Corollary 4 (see theorem 1.4(ii) in [9]). *Let G be a finite group and $\Sigma = \Sigma(\Delta)$, where Δ is the set of all central chief factors of G . Then the lattice $\mathfrak{L}_\Sigma(G)$ is distributive if and only if $\mathfrak{L}_\Sigma(G) = \mathfrak{L}_n(G)$ is distributive.*

Corollary 5 (see theorem 1.2 in [9]). *Let G be a finite group and either $\Sigma = \Sigma(\Delta)$, where Δ is the set of all \mathfrak{F} -central chief factors of G for some class of groups containing all identity groups \mathfrak{F} , or $\Sigma = \Sigma_G(\mathfrak{F})$ for some Fitting formation \mathfrak{F} , then $\mathfrak{L}_\Sigma(G)$ is a sublattice in $\mathfrak{L}(G)$.*

Some further applications

A group is called *primary* if it is a finite p -group for some prime p . If $\sigma = \{\sigma_i \mid i \in I\}$ is any partition of the set of all primes \mathbb{P} , that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$, then we say, following [11], that the group G is:

σ -*primary* if it is a finite σ_i -group for some i ; σ -*soluble* if G is finite and every its chief factor is σ -primary; σ -*nilpotent* or σ -*decomposable* [12] if $G = G_1 \times \dots \times G_n$ for some σ -primary groups G_1, \dots, G_n . Observe that a finite group is primary (respectively soluble, nilpotent) if and only if it is σ -primary (respectively σ -soluble, σ -nilpotent), where $\sigma = \{\{2\}, \{3\}, \dots\}$.

In this section we discuss some applications of the lattice $\mathfrak{L}_\Sigma(G)$ in the theory of finite groups. And we start with one application of the lattices $\mathfrak{L}_{\Sigma_G(\mathfrak{N}_\sigma)}(G)$ and $\mathfrak{L}_{\Sigma_G(\Delta)}(G)$, where \mathfrak{N}_σ is the class of all σ -nilpotent groups and Δ is the set of all σ -central, that is, \mathfrak{N}_σ -central chief factors of G , in the theory of generalized T -groups.

Lattice characterizations of finite σ -soluble $P\sigma T$ -groups. We say, following [11], that the subgroup A of G is σ -*subnormal* in G if it is \mathfrak{N}_σ -*subnormal* in G in the sense of Kegel. Note that a subgroup A of G is subnormal in G if and only if A is σ -subnormal in G , where $\sigma = \{\{2\}, \{3\}, \dots\}$.

A subgroup A of a finite group G is said to be: *quasinormal* (respectively *S-quasinormal* or *S-permutable* [13]) in G if A permutes with all subgroups (respectively with all Sylow subgroups) H of G , that is, $AH = HA$; σ -*permutable* in G [11] if A permutes with all Hall σ_i -subgroups of G for all i .

Recall that a finite group G is said to be a T -*group* (respectively PT -*group*, PST -*group*) if every subnormal subgroup of G is normal (respectively permutable, S -permutable) in G ; G is said to be a $P\sigma T$ -*group* if every σ -subnormal subgroup of G is σ -permutable in G .

The description of PST -groups, that are groups, in which every subnormal subgroup is S -permutable, was first obtained by Agrawal [14], for the soluble case, and by Robinson in [15], for the general case. In the further publications, authors (see, for example, the recent papers [16–25]) have found out and described many other interesting characterizations of soluble PST -groups. Some characterizations of $P\sigma T$ -groups were obtained in the papers [11; 26]. Theorem 2.4 allows to prove the following result in this line research.

Theorem 4. *Suppose that G is a finite σ -soluble group. Then G is a $P\sigma T$ -group if and only if $\mathfrak{L}_{\Sigma_G(\mathfrak{N}_\sigma)}(G) = \mathfrak{L}_{\Sigma_G(\Delta)}(G)$, where Δ is the set of all σ -central chief factors of G .*

The proof of theorem 4 consists of many steps and it uses theorems 1 and 2 and also the following lemmas.

Lemma 3. *Let \mathfrak{F} be a class of groups, N be a normal subgroup of G and Σ be a formation Fitting set of G .*

(1) *If $\Sigma = \Sigma_G(\Delta)$, where Δ is the set of all \mathfrak{F} -central chief factors of G , then $\Sigma N/N = \Sigma_{G/N}(\Delta^*)$, where Δ^* is the set of all \mathfrak{F} -central chief factors of G/N .*

(2) $\Sigma_G(\mathfrak{F})N/N = \Sigma_{G/N}(\mathfrak{F})$.

Proof. (1) This follows from proposition (iv) and the fact that a chief factor $(H/N)/(K/N)$ is \mathfrak{F} -central in G/N if and only if the chief factor H/K is \mathfrak{F} -central in G (see lemma 2(1)).

(2) This follows from proposition (iv).

The lemma is proved.

Lemma 4. Let Σ be a formation Fitting set of G and let $A \in \mathfrak{L}_\Sigma(G)$ and $N \leq H \leq G$, where $N \trianglelefteq G$:

- (1) $AN/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$;
- (2) if $H/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$, then $H \in \mathfrak{L}_\Sigma(G)$;
- (3) $A \cap E \in \mathfrak{L}_{\text{formfit}(\Sigma \cap E)}(E)$ for every subgroup E of G .

Proof. (1) Since $A \in \mathfrak{L}_\Sigma(G)$, $A^G/A_G \in \Sigma$ and so

$$(A^G N/N)/(A_G N/N) \in \Sigma N/N.$$

On the other hand, we have that

$$(AN/N)^{G/N} = (AN)^G/N = A^G N/N,$$

where $A_G N/N \leq (AN/N)_{G/N}$. Hence

$$(AN/N)^{G/N}/(AN/N)_{G/N} \in \Sigma N/N$$

since $\Sigma N/N$ is a formation Fitting set of G/N by proposition (iv), so $AN/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$.

(2) Since $H/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$, we have

$$(H^G/N)/(H_G/N) = (H/N)^{G/N}/(H/N)_{G/N} \in \Sigma N/N$$

and so $H^G/H_G \in \Sigma$ by proposition (i). Hence $H \in \mathfrak{L}_\Sigma(G)$.

(3) Let $\Sigma_0 = \text{formfit}(\Sigma \cap E)$. It is clear that

$$(A^G \cap E)/(A_G \cap E) \in \Sigma \cap E \subseteq \Sigma_0.$$

On the other hand, we have

$$A_G \cap E \leq (A \cap E)_E \leq A \cap E \leq (A \cap E)^E \leq A^G \cap E$$

and so $(A \cap E)^E/(A \cap E)_E \in \Sigma_0$ since Σ_0 is a formation Fitting set of E . Hence $A \cap E \in \mathfrak{L}_{\Sigma_0}(E)$.

The lemma is proved.

Lemma 5. Let \mathfrak{F} be a saturated formation and G be a finite group:

- (1) if $G \in \mathfrak{F}$, then every chief factor of G is \mathfrak{F} -central in G ;
- (2) if G has a normal subgroup N with $G/N \in \mathfrak{F}$ such that every chief factor of G below N is \mathfrak{F} -central in G , then $G \in \mathfrak{F}$.

Proof. (1) This part directly follows from the Barnes – Kegel result [6, chapter IV, proposition 1.5].

(2) In fact, in view of part (1) and the Jordan – Hölder's theorem for the chief series, it is enough to show that if every chief factor of G is \mathfrak{F} -central in G , then $G \in \mathfrak{F}$. Assume that this is false and let G be a counterexample of minimal order. Then G has a unique minimal normal subgroup, R say, and $R \not\leq \Phi(G)$. Moreover, R is abelian since otherwise we have $G \simeq G/C_G(R) = G/1 \in \mathfrak{F}$. Hence $R = C_G(R)$ by [6, chapter A, theorem 15.6] and for some maximal subgroup M of G we have $G = R \rtimes M$. Therefore the map

$$f: G \rightarrow R \rtimes (G/C_G(R)) = R \rtimes (G/R)$$

with $f(rm) = (r, mR)$ for all $r \in R$ and $m \in M$ is isomorphism, so $G \in \mathfrak{F}$ since the factor $R/1$ is \mathfrak{F} -central in G by hypothesis.

The lemma is proved.

Recall that the σ -nilpotent residual G^{σ_0} of a finite groups G is the intersection of all normal subgroups N of G with σ -nilpotent quotient G/N .

Lemma 6 (see theorem A in [26]). Let $D = G^{\sigma_0}$ be the σ -nilpotent residual of a finite group G . If G is σ -soluble P σ T-group, then the following conditions hold:

- (1) $G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ -nilpotent and every element of G induces a power automorphism in D ;
- (2) $O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i -subgroup of G for all i .

Conversely, if conditions (1) and (2) hold for some subgroups D and M of G , then G is a P σ T-group.

Lemma 7. Let N be a normal subgroup of a finite group G such that every chief factor of G below N is G -central in G . Then N is σ -nilpotent, and if N is a σ_i -group, then $O^{\sigma_i}(G) \leq C_G(N)$.

Proof. Let $1 = Z_0 < Z_1 < \dots < Z_t = N$ be a chief series of G below N and $C_i = C_G(Z_i/Z_{i-1})$. First we show that N is σ -nilpotent. By hypothesis, Z_1 and G/G_1 are σ_j -groups for some j . Now let H/K be any chief factor of N such that $H \leq Z_1$. From the isomorphism $C_1 N/N \cong N/(C_1 \cap N)$ it follows that H/K and $N/C_N(H/K)$ are σ_j -groups. Therefore every chief factor of N below Z_1 is N_σ -central in N . On the other hand, N/Z_1 is σ -nilpotent by induction and so N is σ -nilpotent by lemma 5, condition (2).

Finally, assume that N is a σ_i -group and let $C = C_1 \cap \dots \cap C_t$. Then G/C is a σ_i -group. On the other hand, $C/C_G(N) \cong A \leq \text{Aut}(N)$ stabilizes the series $1 = Z_0 < Z_1 < \dots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$ -group by [6, chapter A, corollary 12.4]. Hence $C/C_G(N)$ is a σ_i -group, so $O^{\sigma_i}(G) \leq C_G(N)$. The lemma is proved.

Now consider some applications of theorem 4.

Recall that $Z_\sigma(G)$ denotes the σ -hypercentre of G [11], that is, the largest normal subgroup of G such that every chief factor of G below $Z_\sigma(G)$ is σ -central in G . We say, following [13, p. 20], that a subgroup H of a finite group G is σ -hypercentrally embedded in G if $H/H_G \leq Z_\sigma(G/H_G)$ and hypercentrally embedded in G if $H/H_G \leq Z_\infty(G/H_G)$.

Corollary 6 (see theorem 4.1 in [11]). *Let G be a finite σ -soluble group. If every σ -subnormal subgroup of G is σ -hypercentrally embedded in G , then G is a PST-group.*

In the case where $\sigma = \{\{2\}, \{3\}, \dots\}$ we get from theorem 3.1 the following known characterization of finite soluble PST-groups.

Corollary 7 (see theorem 1.3 in [10]). *Suppose that G is a finite soluble group. Then G is a PST-group if and only if $\mathfrak{L}_{\Sigma_G(\mathfrak{N})}(G) = \mathfrak{L}_{\Sigma(\Delta)}(G)$, where Δ is the set of all central chief factors H/K of G , that is, $C_G(H/K) = G$.*

Corollary 8 (see theorem 2.4.4 in [13]). *Let G be a finite group. G is a soluble PST-group if and only if every subnormal subgroup H of G is hypercentrally embedded in G (that is $H/H_G \leq Z_\infty(G/H_G)$).*

Groups with Σ -normal and Σ -abnormal subgroups. Let Σ be a formation Fitting set of G . Then we say that a subgroup A of G is: (i) Σ -normal in G if $A \in \mathfrak{L}_\Sigma(G)$; (ii) Σ -abnormal in G provided $H \notin \mathfrak{L}_{\text{formfit}(\Sigma \cap E)}(E)$ for all subgroups $H < E$ of G , where $A \leq H$.

Example 2. (i) A subgroup A of G is normal in G if and only if it is Σ -normal in G , where $\Sigma = \{H/H \mid H \trianglelefteq G\}$.

(ii) A subgroup A of G is called *abnormal* in G if $g \in \langle A, A^g \rangle$ for all $g \in G$. If G is a soluble finite group, then A is abnormal in G if and only if A is Σ -abnormal in G , where $\Sigma = \Sigma_G(\mathfrak{N})$, by [12, chapter IV, theorem 1.7.1].

(iii) Let Δ be the set of all \mathfrak{F} -central chief factors of G and $\Sigma = \Sigma_G(\Delta)$. If G is finite, then a subgroup A of G is called: (a) \mathfrak{F} -normal in G [8] if $A^G/A_G \in \Sigma$, (b) \mathfrak{F} -abnormal in G [8] if H is not \mathfrak{F} -normal in E for every two subgroups $H < E$ of G such that $A \leq H$. Therefore a subgroup A of G is \mathfrak{F} -normal (\mathfrak{F} -abnormal) in G if and only if it is Σ -normal (respectively Σ -abnormal) in G , where $\Sigma = \Sigma_G(\Delta)$.

(iv) Let G be finite. If A is σ -hypercentrally embedded in G , that is, $A/A_G \leq Z_\sigma(G/A_G)$, then $A^G/A_G \leq Z_\sigma(G/A_G)$. In particular, if A is hypercentrally embedded in G , then $A^G/A_G \leq Z_\infty(G/A_G)$. Therefore A is σ -hypercentrally (hypercentrally) embedded in G if and only if it is Σ -abnormal in G , where $\Sigma = \Sigma_G(\Delta)$ and Δ is the set of all σ -central (respectively central) chief factors of G .

Recall that a finite group G is a *DM-group* [8] if $G = D \rtimes M$ and the following conditions hold: (1) $D = G' \neq 1$ is abelian; (2) $M = \langle x \rangle$ is a cyclic abnormal Sylow p -subgroup of G , where p is the smallest prime dividing $|G|$; (3) $M_G = \langle x^p \rangle = Z(G)$; (4) x induces a fixed-point-free power automorphism on D .

In the paper [27], Fattahi defined *B-groups* to be a finite groups in which every subgroup is either normal or abnormal and he showed that a non-nilpotent finite group G is a *B-group* if and only if G is a *DM-group*. As a generalization of this result, Ebert and Bauman classified the group in which every subgroup is either subnormal or abnormal [28]. In further, the results in [27] have been developed in many other directions (see, for example, the recent papers [8; 29–33]).

We say that G is a *ΣNA -group* if every subgroup of G is either Σ -normal or Σ -abnormal in G for some formation Fitting set Σ of G .

The results in [8; 27–33] and also many other known results of this type are the motivation for the following question.

Question 1. Let Σ be a formation Fitting set of a finite group G . What we can say about the structure of G in the case when at least one of the following conditions holds: (i) every subgroup of G is Σ -normal in G ; (ii) G is a ΣNA -group, where $\Sigma = \Sigma_G(\Delta)$ for some G -closed set Δ of chief factors of G or $\Sigma = \Sigma_G(\mathfrak{F})$ for some hereditary (in the sense of Mal'cev [34]) Fitting formation \mathfrak{F} ?

Note that the answer to question 1 for some special Σ is known. Let, for example, $\Sigma = \{H/H \mid H \trianglelefteq G\}$. Then: (i) every subgroup of G is Σ -normal in G if and only if G is a Dedekind group; (ii) G is a ΣNA -group if and only if G is a P -group by example 2(i) and 2(ii) since every P -group is clearly soluble.

Now let Δ be the set of all \mathfrak{F} -central chief factors of a finite group G and $\Sigma = \Sigma_G(\Delta)$, where \mathfrak{F} is a hereditary saturated formation containing all nilpotent groups. Then G is a ΣNA -group if and only if every subgroup of G is either \mathfrak{F} -normal or \mathfrak{F} -abnormal in G by example 2(iii). Such a class of finite groups is also known.

Theorem 5 (see theorem 1.4 in [8]). *Let \mathfrak{F} be a hereditary saturated formation containing all nilpotent groups. If every subgroup of a finite group G is either \mathfrak{F} -normal or \mathfrak{F} -abnormal in G , then G is of either of the following types:*

- (I) $G \in \mathfrak{F}$;
 - (II) $G = D \rtimes M$ is a DM -group, where $D = G^{\mathfrak{F}}$, and M is an \mathfrak{F} -abnormal subgroup of G with $M_G = Z_{\mathfrak{F}}(G)$.
- Conversely, in a group G of type (I) or (II) every subgroup is either \mathfrak{F} -normal or \mathfrak{F} -abnormal.*

In this theorem $Z_{\mathfrak{F}}(G)$ denotes the \mathfrak{F} -hypercentre of G , that is the product of all normal subgroups N of G such that either $N = 1$ or $N \neq 1$ and every chief factor of G below N is \mathfrak{F} -central in G .

Finite groups G with modular lattices $\mathfrak{L}_{\Sigma}(G)$ and $\mathfrak{L}_{sn}(G)$. A subgroup A of G is called: *subnormal* in G if there exists a subgroup series $A = A_0 \trianglelefteq A_1 \trianglelefteq \dots \trianglelefteq A_{t-1} \trianglelefteq A_t = G$ (*); *composition* in G if every factor A_i/A_{i-1} of the series (*) is a simple group. Note that a subgroup A of a finite group G is subnormal in G if and only if it is composition in G .

Now let Σ be a formation Fitting set of G . We say a subgroup A of G is Σ -subnormal in G if there exists a subgroup series $A = A_0 \trianglelefteq A_1 \trianglelefteq \dots \trianglelefteq A_{t-1} \trianglelefteq A_t = G$ of G such that A_{i-1} is Σ_i -normal in A_i , where $\Sigma_i = \text{formfit}(\Sigma \cap A_i)$, for all $i = 1, \dots, t$.

By classical Wielandt's result [35, theorem 1.1.5], the set $\mathfrak{L}_{sn}(G)$ of all composition subgroups of G forms a sublattice of $\mathfrak{L}(G)$.

Question 2. Let G be finite. For which conditions on the formation Fitting set Σ of G the set of all Σ -subnormal subgroups of G forms a sublattice of $\mathfrak{L}(G)$?

In some special cases the answer to question 2 is known. Indeed, $\mathfrak{L}_n(G) = \mathfrak{L}_{\Sigma}(G)$, where $\Sigma = \{H/H \mid H \trianglelefteq G\}$, is modular. In the paper [9] the following result in this direction was obtained.

Theorem 6 (see theorem 1.4 in [9]). *Let G be finite and $\Sigma = \Sigma_G(\Delta)$, where Δ is the set of all central chief factors of G . Then the lattice $\mathfrak{L}_{\Sigma}(G)$ is modular if and only if every two subgroups $A, B \in \mathfrak{L}_{\Sigma}(G)$ are permutable, that is $AB = BA$.*

Zappa, in his paper [36], described conditions under which the lattice $\mathfrak{L}_{sn}(G)$, where G is finite, is modular.

Theorem 7 (see theorem 9.2.3 in [35]). *The following properties of the finite group G are equivalent:*

- (a) the lattice $\mathfrak{L}_{sn}(G)$ is modular;
- (b) if $T \trianglelefteq S$, where S is subnormal in G and S/T is a p -group, p a prime, then $\mathfrak{L}(S/T)$ is modular;
- (c) if $T \trianglelefteq S$, where S is subnormal in G and $|S/T| = p^3$, p a prime, then $\mathfrak{L}(S/T)$ is modular.

A new characterization of finite groups with modular lattice of the subnormal subgroups was given in the paper [9].

Theorem 8 (see theorem 1.3 in [9]). *Let G be a finite group. Then the lattice $\mathfrak{L}_{sn}(G)$ is modular if and only if for every two subnormal subgroups $L \leq T$ of G , where $L \in \mathfrak{L}_{\Sigma}(T)$ and $\Sigma = \Sigma_T(\mathfrak{N}^*)$, L permutes with every subnormal subgroup M of T .*

Finite groups factorized by Σ -normal subgroups. It is well-known that the product $G = AB$ of two normal finite supersoluble groups A and B is not supersoluble in general. Nevertheless, such a product is supersoluble if the indices $|G:A|$ and $|G:B|$ are coprime [37, chapter 4, theorem 3.4]. Moreover, by Doerk's result [38], the finite group G is supersoluble if it has four supersoluble subgroups A_1, A_2, A_3, A_4 whose indices $|G:A_1|, |G:A_2|, |G:A_3|, |G:A_4|$ are pairwise coprime. In this paper, we prove the following result in this line research.

Theorem 9. Suppose that G is finite and let Δ is the set of all cyclic chief factors of G and $\Sigma = \Sigma_G(\Delta)$. Then G is supersoluble if and only if G has three Σ -normal supersoluble subgroups A_1, A_2, A_3 whose indices $|G : A_1|, |G : A_2|, |G : A_3|$ are pair coprime.

Lemma 8 (see lemma 4.5 in [6, chapter IV]). Let G be a finite group in \mathfrak{F} , where \mathfrak{F} is a saturated Fitting formation and let $p \in \pi(G)$. If $X = G/O_{p',p}(G)$ and R is an irreducible $\mathbb{F}_p X$ -module, then $R \rtimes X \in \mathfrak{F}$.

Proof of theorem 9. We need only to show that the sufficiency of the condition of the theorem holds. Assume that this is false and let G be a counterexample of minimal order. Then $G \neq A_i \neq 1$ for all i and G is soluble by Wielandt's theorem [6, chapter I, theorem 3.4]. Moreover, from $(|G : A_i|, |G : A_j|) = 1$ for $i \neq j$ it follows that $G = A_1 A_2 = A_1 A_3 = A_2 A_3$.

Let R be a minimal normal subgroup of G . Then R is a p -group for some prime p . Note also that $\Sigma R/R = \Sigma_{G/R}(\Delta^*)$, where Δ^* is the set of all cyclic chief factors of G/R by lemma 3(1). On the other hand, the subgroup $A_i R/R$ belongs the lattice $\mathcal{L}_{\Sigma R/R}(G)$ by lemma 4(1), so $A_i R/R \in \mathcal{L}_{\Sigma_{G/R}(\Delta^*)}(G/R)$. Note also that $A_i R/R \simeq A_i / (A_i \cap R)$ is supersoluble. Therefore the hypothesis holds for G/R . Hence G/R is supersoluble, so R is the unique minimal normal subgroup of G and $R \not\leq \Phi(G)$. Thus $R = C_G(R) = O_p(G)$ for some prime p by [6, chapter A, theorem 15.6]. Let G_p be a Sylow p -subgroup of G .

From the hypothesis it follows that for some $i \neq j$ and some $x, y \in G$ we have $R \leq G_p^x \leq A_i$ and $R \leq G_p^y \leq A_j$. Since $R = C_G(R)$, $F(A_i) = O_p(A_i)$. On the other hand, A_i is supersoluble and so $A_i/F(A_i) = A_i/O_p(A_i)$ is abelian. Hence $A_i \leq N_G(G_p^x)$. It follows that $A_i^{x^{-1}} \leq N_G(G_p)$. Similarly, $A_j^{y^{-1}} \leq N_G(G_p)$. Then

$$G = A_i A_j = A_i^{x^{-1}} A_j^{y^{-1}} \leq N_G(G_p)$$

and so

$$R = O_p(G) = G_p = O_p(A_i) = O_p(A_j).$$

Now we show that $R \leq A_k$, where $j \neq k \neq i$. Assume that $R \not\leq A_k$. Then $(A_k)_G = 1$ and $A_k^G \neq 1$ since $A_k \neq 1$. Hence $R \leq A_k^G$, which implies that $R/1$ is cyclic and so G is supersoluble. This contradiction shows that $R \leq A_3$, so $R = G_p = O_p(A_k) = F(A_k)$.

Therefore $A_1 R/R, A_2 R/R, A_3 R/R$ are abelian subgroup of G/R whose indices

$$|G/R : A_1 R/R|, |G/R : A_2 R/R|, |G/R : A_3 R/R|$$

are pair coprime, so G/R is nilpotent by Kegel's theorem [39]. Moreover, for every Sylow subgroup Q/R of G/R we have that $Q/R \leq A_i/R$ or $Q/R \leq A_j/R$. Hence for some subgroups $A/R \leq A_i/R$ and $B/R \leq A_j/R$ we have $G/R = (A/R) \times (B/R)$. It is clear that the subgroups A and B are supersoluble and so the group $A \times B$ is supersoluble. It is clear also that $O_{p',p}(A) = R = O_{p',p}(B)$. Hence

$$X = (A \times B)/O_{p',p}(A \times B) \simeq (A/R) \times (B/R) \simeq G/R.$$

But then G is supersoluble by lemma 8. This contradiction completes the proof of the result.

A subgroup M of G is called *modular* in G if M is a modular element (in the sense of Kurosh [35, p. 43]) of the lattice $\mathcal{L}(G)$. It is known that [35, theorem 5.2.3] for every modular subgroup A of G all chief factors of G between A_G and A^G are cyclic. Therefore we get from theorem 9 the following result.

Corollary 9. If G is finite and G has three modular supersoluble subgroups A_1, A_2, A_3 whose indices $|G : A_1|, |G : A_2|, |G : A_3|$ are pair coprime, then G is supersoluble.

Библиографические ссылки

1. Wielandt H. Eine Verallgemeinerung der invarianten Untergruppen. *Mathematische Zeitschrift*. 1939;45(1):209–244. DOI: 10.1007/BF01580283.
2. Kegel OH. Untergruppenverbände endlicher Gruppen, die Subnormalteilverband echt enthalten. *Archiv der Mathematik*. 1978; 30(1):225–228. DOI: 10.1007/BF01226043.
3. Ballester-Bolinches A, Ezquerro LM. *Classes of Finite Groups*. Dordrecht: Springer; 2006. 381 p. (Mathematics and its applications; volume 584). DOI: 10.1007/1-4020-4719-3.

4. Ballester-Bolinches A, Doerk K, Pérez-Ramos MD. On the lattice of \mathfrak{F} -subnormal subgroups. *Journal of Algebra*. 1992;148(1): 42–52. DOI: 10.1016/0021-8693(92)90235-E.
5. Васильев АФ, Каморников СФ, Семенчук ВН. О решетках подгрупп конечных групп. В: *Бесконечные группы и при-
мыкающие алгебраические структуры*. Киев: Институт математики АН Украины; 1993. с. 27–54.
6. Doerk K, Hawkes T. *Finite soluble groups*. Berlin: Walter de Gruyter; 1992. 910 p. (de Gruyter expositions in mathematics; book 4).
7. Шеметков ЛА, Скиба АН. *Формации алгебраических систем*. Москва: Наука; 1989. 256 с.
8. Hu B, Huang J, Skiba AN. Finite groups with only \mathfrak{F} -normal and \mathfrak{F} -abnormal subgroups. *Journal of Group Theory*. 2019;22: 915–926. DOI: 10.1515/jgth-2018-0199.
9. Chi Z, Skiba AN. On two sublattices of the subgroup lattice of a finite group. *Journal of Group Theory*. 2019;22(6):1035–1047. DOI: 10.1515/jgth-2019-0039.
10. Chi Z, Skiba AN. On a lattice characterization of finite soluble *PST*-groups. *Bulletin of the Australian Mathematical Society*. 2019;99(3):1–8. DOI: 10.1017/S0004972719000741.
11. Skiba AN. On σ -subnormal and σ -permutable subgroups of finite groups. *Journal of Algebra*. 2015;436:1–16. DOI: 10.1016/j.jalgebra.2015.04.010.
12. Шеметков ЛА. *Формации конечных групп*. Москва: Наука; 1978. 272 с.
13. Ballester-Bolinches A, Esteban-Romero R, Asaad M. *Products of Finite Groups*. Berlin: Walter de Gruyter; 2010. (de Gruyter expositions in mathematics; volume 53). DOI: 10.1515/9783110220612.
14. Agrawal RK. Finite groups whose subnormal subgroups permute with all Sylow subgroups. *Proceedings of the American Mathematical Society*. 1975;47:77–83. DOI: 10.1090/S0002-9939-1975-0364444-4.
15. Robinson DJS. The structure of finite groups in which permutability is a transitive relation. *Journal of the Australian Mathematical Society*. 2001;70(2):143–160. DOI: 10.1017/S1446788700002573.
16. Brice RA, Cossey J. The Wielandt subgroup of a finite soluble groups. *Journal of the London Mathematical Society*. 1989; 40(2):244–256. DOI: 10.1112/jlms/s2-40.2.244.
17. Beidleman JC, Brewster B, Robinson DJS. Criteria for permutability to be transitive in finite groups. *Journal of Algebra*. 1999; 222(2):400–412. DOI: 10.1006/jabr.1998.7964.
18. Ballester-Bolinches A, Esteban-Romero R. Sylow permutable subnormal subgroups of finite groups. *Journal of Algebra*. 2002; 251(2):727–738. DOI: 10.1006/jabr.2001.9138.
19. Ballester-Bolinches A, Beidleman JC, Heineken H. Groups in which Sylow subgroups and subnormal subgroups permute. *Illinois Journal of Mathematics*. 2003;47(1–2):63–69. DOI: 10.1215/ijm/1258488138.
20. Ballester-Bolinches A, Beidleman JC, Heineken H. A local approach to certain classes of finite groups. *Communications in Algebra*. 2003;31(12):5931–5942. DOI: 10.1081/AGB-120024860.
21. Asaad M. Finite groups in which normality or quasnormality is transitive. *Archiv der Mathematik*. 2004;83(4):289–296. DOI: 10.1007/s00013-004-1065-4.
22. Ballester-Bolinches A, Cossey J. Totally permutable products of finite groups satisfying *SC* or *PST*. *Monatshefte für Mathematik*. 2005;145(2):89–94. DOI: 10.1007/s00605-004-0263-9.
23. Al-Sharo KA, Beidleman JC, Heineken H, Ragland MF. Some characterizations of finite groups in which semipermutability is a transitive relation. *Forum Mathematicum*. 2010;22(5):855–862. DOI: 10.1515/forum.2010.045.
24. Beidleman JC, Ragland MF. Subnormal, permutable, and embedded subgroups in finite groups. *Central European Journal of Mathematics*. 2011;9(4):915–921. DOI: 10.2478/s11533-011-0098-8.
25. Yi X, Skiba AN. Some new characterizations of *PST*-groups. *Journal of Algebra*. 2014;399:39–54. DOI: 10.1016/j.jalgebra.2013.10.001.
26. Skiba AN. Some characterizations of finite σ -soluble *PST*-groups. *Journal of Algebra*. 2018;495:114–129. DOI: 10.1016/j.jalgebra.2017.11.009.
27. Fattahi A. Groups with only normal and abnormal subgroups. *Journal of Algebra*. 1974;28(1):15–19. DOI: 10.1016/0021-8693(74)90019-2.
28. Ebert G, Bauman S. A note on subnormal and abnormal chains. *Journal of Algebra*. 1975;36(2):287–293. DOI: 10.1016/0021-8693(75)90103-9.
29. Semenchuk VN, Skiba AN. On one generalization of finite \mathfrak{U} -critical groups. *Journal of Algebra and its Applications*. 2016; 15(4):1650063. DOI: 10.1142/S0219498816500638.
30. Монахов ВС. Конечные группы с абнормальными и \mathfrak{U} -субнормальными подгруппами. *Сибирский математический журнал*. 2016;57(2):447–462. DOI: 10.17377/smzh.2016.57.217.
31. Монахов ВС, Сохор ИЛ. Конечные группы с формационно субнормальными примарными подгруппами. *Сибирский математический журнал*. 2017;58(4):851–863. DOI: 10.17377/smzh.2017.58.412.
32. Monakhov VS, Sokhor IL. On groups with formational subnormal Sylow subgroups. *Journal of Group Theory*. 2018;21:273–287. DOI: 10.1515/jgth-2017-0039.
33. Monakhov VS, Sokhor IL. Finite groups with abnormal or formational subnormal primary subgroups. *Communications in Algebra*. 2019;47(10):3941–3949. DOI: 10.1080/00927872.2019.1572174.
34. Мальцев АИ. *Алгебраические системы*. Москва: Наука; 1970. 392 с.
35. Schmidt R. *Subgroup lattices of groups*. Berlin: Walter de Gruyter; 1994. 572 p. (de Gruyter expositions of mathematics; volume 14).
36. Zappa G. Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione è modulare. *Bollettino dell'Unione Matematica Italiana. Serie 3*. 1956;11(3):315–318.
37. Bray HB. *Between nilpotent and solvable*. Weinstein M, editor. Passaic: Polygonal Publishing House; 1982. 231 p.
38. Doerk K. Minimal nicht überauflösbare, endlicher Gruppen. *Mathematische Zeitschrift*. 1966;91(3):198–205. DOI: 10.1007/BF01312426.
39. Kegel OH. Zur Struktur mehrfach faktorisierbarer endlicher Gruppen. *Mathematische Zeitschrift*. 1965;87(1):42–48. DOI: 10.1007/BF01109929.

References

1. Wielandt H. Eine Verallgemeinerung der invarianten Untergruppen. *Mathematische Zeitschrift*. 1939;45(1):209–244. DOI: 10.1007/BF01580283.
2. Kegel OH. Untergruppenverbände endlicher Gruppen, die Subnormalteilerverband echt enthalten. *Archiv der Mathematik*. 1978; 30(1):225–228. DOI: 10.1007/BF01226043.
3. Ballester-Bolinches A, Ezquerro LM. *Classes of Finite Groups*. Dordrecht: Springer; 2006. 381 p. (Mathematics and its applications; volume 584). DOI: 10.1007/1-4020-4719-3.
4. Ballester-Bolinches A, Doerk K, Pérez-Ramos MD. On the lattice of \mathfrak{F} -subnormal subgroups. *Journal of Algebra*. 1992;148(1): 42–52. DOI: 10.1016/0021-8693(92)90235-E.
5. Vasil'ev AF, Kamornikov SF, Semenchuk VN. [On lattices of subgroups of finite groups]. In: *Beskonechnye gruppy i primykayushchie algebraicheskie struktury* [Infinite groups and related algebraic structures]. Kiev: Institute of Mathematics of National Academy of Sciences of Ukraine; 1993. p. 27–54. Russian.
6. Doerk K, Hawkes T. *Finite soluble groups*. Berlin: Walter de Gruyter; 1992. 910 p. (de Gruyter expositions in mathematics; book 4).
7. Shemetkov LA, Skiba AN. *Formatsii algebraicheskikh sistem* [Formations of algebraic systems]. Moscow: Nauka; 1989. 256 p. Russian.
8. Hu B, Huang J, Skiba AN. Finite groups with only \mathfrak{F} -normal and \mathfrak{F} -abnormal subgroups. *Journal of Group Theory*. 2019;22: 915–926. DOI: 10.1515/jgth-2018-0199.
9. Chi Z, Skiba AN. On two sublattices of the subgroup lattice of a finite group. *Journal of Group Theory*. 2019;22(6):1035–1047. DOI: 10.1515/jgth-2019-0039.
10. Chi Z, Skiba AN. On a lattice characterization of finite soluble *PST*-groups. *Bulletin of the Australian Mathematical Society*. 2019;99(3):1–8. DOI: 10.1017/S0004972719000741.
11. Skiba AN. On σ -subnormal and σ -permutable subgroups of finite groups. *Journal of Algebra*. 2015;436:1–16. DOI: 10.1016/j.jalgebra.2015.04.010.
12. Shemetkov LA. *Formatsii konechnykh grupp* [Formations of finite groups]. Moscow: Nauka; 1978. 272 p. Russian.
13. Ballester-Bolinches A, Esteban-Romero R, Asaad M. *Products of Finite Groups*. Berlin: Walter de Gruyter; 2010. (de Gruyter expositions in mathematics; volume 53). DOI: 10.1515/9783110220612.
14. Agrawal RK. Finite groups whose subnormal subgroups permute with all Sylow subgroups. *Proceedings of the American Mathematical Society*. 1975;47:77–83. DOI: 10.1090/S0002-9939-1975-0364444-4.
15. Robinson DJS. The structure of finite groups in which permutability is a transitive relation. *Journal of the Australian Mathematical Society*. 2001;70(2):143–160. DOI: 10.1017/S1446788700002573.
16. Brice RA, Cossey J. The Wielandt subgroup of a finite soluble groups. *Journal of the London Mathematical Society*. 1989; 40(2):244–256. DOI: 10.1112/jlms/s2-40.2.244.
17. Beidleman JC, Brewster B, Robinson DJS. Criteria for permutability to be transitive in finite groups. *Journal of Algebra*. 1999; 222(2):400–412. DOI: 10.1006/jabr.1998.7964.
18. Ballester-Bolinches A, Esteban-Romero R. Sylow permutable subnormal subgroups of finite groups. *Journal of Algebra*. 2002; 251(2):727–738. DOI: 10.1006/jabr.2001.9138.
19. Ballester-Bolinches A, Beidleman JC, Heineken H. Groups in which Sylow subgroups and subnormal subgroups permute. *Illinois Journal of Mathematics*. 2003;47(1–2):63–69. DOI: 10.1215/ijm/1258488138.
20. Ballester-Bolinches A, Beidleman JC, Heineken H. A local approach to certain classes of finite groups. *Communications in Algebra*. 2003;31(12):5931–5942. DOI: 10.1081/AGB-120024860.
21. Asaad M. Finite groups in which normality or quasnormality is transitive. *Archiv der Mathematik*. 2004;83(4):289–296. DOI: 10.1007/s00013-004-1065-4.
22. Ballester-Bolinches A, Cossey J. Totally permutable products of finite groups satisfying *SC* or *PST*. *Monatshefte für Mathematik*. 2005;145(2):89–94. DOI: 10.1007/s00605-004-0263-9.
23. Al-Sharo KA, Beidleman JC, Heineken H, Ragland MF. Some characterizations of finite groups in which semipermutability is a transitive relation. *Forum Mathematicum*. 2010;22(5):855–862. DOI: 10.1515/forum.2010.045.
24. Beidleman JC, Ragland MF. Subnormal, permutable, and embedded subgroups in finite groups. *Central European Journal of Mathematics*. 2011;9(4):915–921. DOI: 10.2478/s11533-011-0098-8.
25. Yi X, Skiba AN. Some new characterizations of *PST*-groups. *Journal of Algebra*. 2014;399:39–54. DOI: 10.1016/j.jalgebra.2013.10.001.
26. Skiba AN. Some characterizations of finite σ -soluble $P\sigma T$ -groups. *Journal of Algebra*. 2018;495:114–129. DOI: 10.1016/j.jalgebra.2017.11.009.
27. Fattahi A. Groups with only normal and abnormal subgroups. *Journal of Algebra*. 1974;28(1):15–19. DOI: 10.1016/0021-8693(74)90019-2.
28. Ebert G, Bauman S. A note on subnormal and abnormal chains. *Journal of Algebra*. 1975;36(2):287–293. DOI: 10.1016/0021-8693(75)90103-9.
29. Semenchuk VN, Skiba AN. On one generalization of finite \mathfrak{U} -critical groups. *Journal of Algebra and its Applications*. 2016; 15(4):1650063. DOI: 10.1142/S0219498816500638.
30. Monakhov VS. [Finite groups with abnormal and \mathfrak{U} -subnormal subgroups]. *Sibirskii matematicheskii zhurnal*. 2016;57(2): 447–462. Russian. DOI: 10.17377/smzh.2016.57.217.
31. Monakhov VS, Sokhor IL. [Finite groups with formation subnormal primary subgroups]. *Sibirskii matematicheskii zhurnal*. 2017;58(4):851–863. Russian. DOI: 10.17377/smzh.2017.58.412.
32. Monakhov VS, Sokhor IL. On groups with formational subnormal Sylow subgroups. *Journal of Group Theory*. 2018;21:273–287. DOI: 10.1515/jgth-2017-0039.
33. Monakhov VS, Sokhor IL. Finite groups with abnormal or formational subnormal primary subgroups. *Communications in Algebra*. 2019;47(10):3941–3949. DOI: 10.1080/00927872.2019.1572174.
34. Mal'tsev AI. *Algebraicheskie sistemy* [Algebraic systems]. Moscow: Nauka; 1970. 392 p. Russian.

35. Schmidt R. *Subgroup lattices of groups*. Berlin: Walter de Gruyter; 1994. 572 p. (de Gruyter expositions of mathematics; volume 14).
36. Zappa G. Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione è modulare. *Bollettino dell'Unione Matematica Italiana. Serie 3*. 1956;11(3):315–318.
37. Bray HB. *Between nilpotent and solvable*. Weinstein M, editor. Passaic: Polygonal Publishing House; 1982. 231 p.
38. Doerk K. Minimal nicht überauflösbare, endlicher Gruppen. *Mathematische Zeitschrift*. 1966;91(3):198–205. DOI: 10.1007/BF01312426.
39. Kegel OH. Zur Struktur mehrfach faktorisierbarer endlicher Gruppen. *Mathematische Zeitschrift*. 1965;87(1):42–48. DOI: 10.1007/BF01109929.

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