Вещественный, комплексный и функциональный анализ

Real, complex and functional analysis

ВЛОЖЕНИЕ КЛАССА ХАЙЛАША – СОБОЛЕВА $M_\alpha^p(X)$ В ПРОСТРАНСТВО НЕПРЕРЫВНЫХ ФУНКЦИЙ В КРИТИЧЕСКОМ СЛУЧАЕ

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Пусть (X, d, μ) – метрическое пространство с мерой, удовлетворяющей условию удвоения с показателем γ, т. е. для любых шаров $B(x, R)$ и $B(x, r)$, $r < R$, выполняется неравенство

$$\mu\big(B(x, R)\big) \le a_\mu \left(\frac{R}{r}\right)^\gamma \mu\big(B(x, r)\big)$$

для некоторых положительных постоянных γ и a_μ. На такой общей структуре можно ввести пространство Хайлаша – Соболева $M_\alpha^p(X)$, которое в евклидовом случае совпадает с классическим соболевским пространством при $p > 1$, $\alpha = 1$. В статье обсуждается вложение функций из пространств Хайлаша – Соболева $M_\alpha^p(X)$ в пространство непрерывных функций при $p \le 1$ в критическом случае $\gamma = \alpha p$. Более точно, показано, что любая функция из класса Хайлаша – Соболева $M_\alpha^p(X)$, $0 < p \le 1$, $\alpha > 0$, имеет непрерывного представителя в случае равномерно совершенного пространства (X, d, μ).

Ключевые слова: анализ на метрических пространствах с мерой; пространства Соболева.

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INCLUSION OF HAJŁASZ − SOBOLEV CLASS $M_\alpha^p(X)$ INTO THE SPACE OF CONTINUOUS FUNCTIONS IN THE CRITICAL CASE

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Let (X, d, μ) be a doubling metric measure space with doubling dimension γ, i. e. for any balls $B(x, R)$ and $B(x, r)$, $r < R$, following inequality holds

$$\mu(B(x, R)) \le a_\mu \left(\frac{R}{r}\right)^\gamma \mu(B(x, r))$$

for some positive constants γ and a_μ. Hajłasz – Sobolev space $M_\alpha^p(X)$ can be defined upon such general structure. In the Euclidean case Hajłasz – Sobolev space coincides with classical Sobolev space when $p > 1$, $\alpha = 1$. In this article we discuss inclusion of functions from Hajłasz – Sobolev space $M_\alpha^p(X)$ into the space of continuous functions for $p \le 1$ in the critical case $\gamma = \alpha p$. More precisely, it is shown that any function from Hajłasz – Sobolev class $M_\alpha^p(X)$, $0 < p \le 1$, $\alpha > 0$, has a continuous representative in case of uniformly perfect space (X, d, μ).

Keywords: analysis on metric measure spaces; Sobolev spaces.

Introduction

Denote by W_k^p usual Sobolev space ($p \ge 1$ is a summability parameter, $k \in \mathbb{N}$ is a smoothness parameter). These spaces were introduced by S. L. Sobolev in 1930s. They play crucial role in many areas of mathematics and applications, especially in partial differential equations. Such a big variety of applications is due to the fact that Sobolev spaces allow to define the smoothness of functions in a form suitable for literally any purpose (e. g., weak solutions of many partial differential equations exist in Sobolev spaces). Different generalizations of classical W_k^p space (say, fractional scales of these classes) play the same role. One of the possible generalizations is a Hajłasz – Sobolev class $M_\alpha^p(X)$ which allows us to define Sobolev space on arbitrary metric space X.

Nowadays there is a big interest in so called nonsmooth calculus (see [1]) due to the recent developments in fractals, nonlinear harmonic analysis, analysis on Riemannian manifolds, etc. Many facts of classical analysis can be obtained without strong assumptions on the structure of the underlying space. For example, maximal function, differentiation theorems, functions of bounded mean oscillation can be defined in the context of the space of homogenous type, which will be defined below.

That's why Sobolev classes on general metric measure spaces are of big interest as well. In [2] P. Hajłasz introduced first order Sobolev class $M_1^p(X)$ on arbitrary metric measure space (X, d, μ) for $p > 1$. This approach is based on pointwise inequalities for the pair of functions (f, g)

$$|f(x) - f(y)| \le d(x, y)(g(x) + g(y)) \tag{1}$$

and can be considered as a Lipschitz type characterization of Sobolev functions. One can define Hajłasz – Sobolev space $M_1^p(X)$ as

$$M_1^p(X) = \left\{ f \in L^p(X) : \exists g \in L^p(X) \text{ such that (1) holds a. e.} \right\}.$$

In [2] it was shown that $M_1^p(\mathbb{R}^n) = W_1^p(\mathbb{R}^n)$ for $p > 1$. D. Yang [3] introduced fractional scales of Hajłasz – Sobolev spaces $M_\alpha^p(X)$, $\alpha > 0$.

Let $\alpha, p > 0$. Hajłasz – Sobolev class $M_\alpha^p(X)$ on the metric space X consists of the functions (equivalent classes of functions) $f \in L^p(X)$, such that there exists a nonnegative function $g \in L^p(X)$ and the following inequality

$$|f(x) - f(y)| \le (d(x, y))^\alpha (g(x) + g(y)) \tag{2}$$

takes place almost everywhere. The inequality (2) can be interpreted as a Lipschitz type characterization of Sobolev functions. Any function g which satisfies (2) is called an α-gradient of f. By $D_\alpha[f]$ we denote the set of all α-gradients. In euclidean case $X = \mathbb{R}^n$ one can take maximal function of a gradient $|\nabla f|$ as g.

One can endow space $M_\alpha^p(X)$ with norm (quasinorm for $p < 1$)

$$\|f\|_{M_\alpha^p(X)} = \|f\|_{L^p(X)} + \inf\left\{\|g\|_{L^p(X)}\right\},$$

where infinum is taken over all $g \in L^p(X)$ that satisfy condition (2).

It is well known that functions from Sobolev space $W_1^1[0,1]$ are absolutely continuous. What can one say about this in case of arbitrary metric space? It appears that in critical case (i. e. p is equal to the dimension of the space) any function $f \in M_1^p(X)$, $p \le 1$, has a continuous representative if (X, d, μ) is Ahlfors regular space (see precise definition in the next section). This result was proved by X. Zhou in [4]. The goal of this paper is to generalize this result to Hajłasz – Sobolev spaces $M_\alpha^p(X)$, $\alpha > 0$, and to relax a little bit assumptions on the underlying space: one can obtain the same inclusion with uniformly perfect metric measure space satisfying doubling condition (see precise definitions in the next section). Author came to these questions while investigating Lebesgue points of functions from Hajłasz – Sobolev spaces (see [5] and [6] for subcritical and critical case respectively).

The main result of paper [4] is the following theorem.

Theorem 1. *Let $0 < p \le 1$. Any function $f \in M_1^p(X)$ for p-Ahlfors regular metric measure space (X, d, μ) has a uniformly continuous representative. Moreover, there exists a constant c such that for any ball B and any 1-gradient $g \in D_1[f] \cap L^p(X)$*

$$\sup_{x, y \in B} |f(x) - f(y)| \le c\left(\int_{2B} g^p d\mu\right)^{\frac{1}{p}}.$$

Re-examination of the proof of theorem 1 allows us to state a bit stronger result.

Theorem 2. *Let (X, d, μ) be a uniformly perfect metric measure space which satisfies doubling condition with doubling dimension γ. Let also $\gamma = \alpha p$, $\alpha > 0$ and $0 < p \le 1$. Then any function $f \in M_\alpha^p(X)$ has a continuous representative. Moreover, there exists a constant c such that for any ball B and any α-gradient $g \in D_\alpha[f] \cap L^p(X)$*

$$\sup_{x, y \in B} |f(x) - f(y)| \le cr_B^\alpha\left(\frac{1}{\mu(2B)}\int_{2B} g^p d\mu\right)^{\frac{1}{p}}. \tag{3}$$

It should be noted that expression on the right hand side of (3) tends to zero a. e. according to lemma 2.

In the Euclidean case there always exists a discontinuous function in $W_1^n\left(B\left(0, e^{-1}\right)\right)$ if $n \ge 2$. Indeed, one can take as an example function $\ln|\ln|x\|$ which also shows that $W_1^n(X)$ is not embedded into L^∞ space. The same result is valid for Hajłasz – Sobolev spaces $M_1^p(X)$ as well, provided that (X, d, μ) is p-Ahlfors regular. Existence of discontinuous function from $M_1^p(X)$ was shown by P. Górka and A. Słabuszewski in [7]. Just like in the classical case they observed the function

$$f(x) = \begin{cases} \log\left(-\log d(x, x_0)\right), & x \in B\left(x_0, e^{-1}\right) \setminus \{x_0\}, \\ 0, & \text{otherwise} \end{cases}$$

for some fixed point $x_0 \in X$. It can be shown that $f(x)$ gives an example of discontinuous function in $M_1^p(X)$ [7].

Preliminaries

In this section we collect all necessary definitions and preliminary results.

Let (X, d, μ) be a metric measure space equipped with metric d and σ-finite Borel measure μ. We will assume that the measure satisfies a doubling condition, i. e. for any balls $B(x, r)$ and $B(x, R)$, $R > r$, following inequality holds

$$\mu(B(x, R)) \le a_\mu\left(\frac{R}{r}\right)^\gamma \mu(B(x, r))$$

for some nonnegative constants a_μ and γ. In order to exclude some trivial cases we will also assume that

$$0 < \mu(B) < \infty$$

for any ball B. Then (X, d, μ) is called a space of homogeneous type. Parameter γ plays the role of the dimension of the space. Further we will be working only with measures that satisfy doubling condition.

Definition 1. We will call a metric space *uniformly perfect*, if there exists some constant $0 < \lambda < 1$ such that for any $x \in X$ and any radius $r > 0$

$$B(x, r) \setminus B(x, \lambda r) \neq \varnothing.$$

Definition 2. A metric space is called a *Ahlfors γ-regular space*, if there exist positive constants c_1, c_2 such that for any ball $B(x, r)$

$$c_1 r^\gamma \leq \mu(B(x, r)) \leq c_2 r^\gamma.$$

Important examples of Ahlfors regular spaces are fractals. For example, standard middle third Cantor set, von Koch snowflake curve and Sierpinski triangle equipped with Euclidean distance and corresponding Hausdorff measure are Ahlfors γ-regular spaces with γ equal to $\dfrac{\log 2}{\log 3}$, $\dfrac{\log 4}{\log 3}$ and $\dfrac{\log 3}{\log 2}$ respectively (Hausdorff dimensions of these fractals).

Note that if space is Ahlfors γ-regular then it's necessarily uniformly perfect. The converse is not true.

We use following notation for integral average of function f over the ball B

$$f_B = \fint_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu.$$

Next we recall the definition of Hardy – Littlewood maximal operator Mf:

$$Mf(x) = \sup_{B \ni x} \fint_B |f| \, d\mu,$$

where the supremum is taken over all balls containing point x.

The following lemma shows that maximal function satisfies weak type estimate. One may find the proof in [8] for Euclidean case and in [9, chap. 2] for general metric case. Throughout the paper by c we denote some positive constant whose value is not important and can change even in the same line.

Lemma 1. *Let $f \in L^1(X)$. Then*

$$\mu(\{x \in X : Mf(x) > \lambda\}) \leq c \frac{\|f\|_{L^1(X)}}{\lambda}. \tag{4}$$

Note that (4) does not necessarily hold without doubling condition.

Lemma 2. *Let $0 \leq g \in L^p(X)$. Denote*

$$E = \left\{ x \in X : \limsup_{r \to 0} r^{\alpha p} \fint_{B(x, r)} g^p \, d\mu > 0 \right\}.$$

Then $\mu(E) = 0$.

P r o o f. It's easy to notice that

$$E \subset F = \left\{ x \in X : Mg^p(x) = +\infty \right\}.$$

Indeed, if $Mg^p(x) < \infty$ for some point x then

$$\limsup_{r \to 0} r^{\alpha p} \fint_{B(x, r)} g^p \, d\mu \leq Mg^p(x) \lim_{r \to 0} r^{\alpha p} = 0.$$

It's enough to show that $\mu(F) = 0$. This is a simple consequence of weak type estimate from lemma 1:

$$\mu(F) = \lim_{\lambda \to +\infty} \mu(\{x \in X : Mg^p(x) > \lambda\}) \leq c \lim_{\lambda \to +\infty} \frac{\|g^p\|_{L^1(X)}}{\lambda} = 0.$$

In order to prove main theorem we'll need one more geometric lemma. The proof can be found in [4].

Lemma 3. *Let X be a uniformly perfect space. Then there exists a constant c_0 such that for any ball $B(x, r)$ there exists a sequence of balls $\{B_i\}_{i=1}^\infty = \left\{ B(x_i, c_0^i r) \right\}_{i=1}^\infty$ with following properties:*

1) $B_i \subset B(x, r)$;
2) $B_i \cap B_j = \varnothing$, $i \neq j$;
3) $B_i \subset B(x, c_0^{i-1} r) \setminus B(x, c_0^i r)$.

Proof of the main theorem

We will follow the ideas from [4].

P r o o f of theorem 2. Let $f \in M_\alpha^p(X)$. By definition there exists set $E \subset X$, $\mu(E) = 0$ such that inequality

$$|f(x) - f(y)| \leq (d(x, y))^\alpha (g(x) + g(y))$$

holds for any $x, y \in X \setminus E$. Denote also

$$F = \left\{ x \in X \setminus E : g(x) < \infty \text{ and } \limsup_{r \to +0} r^{\alpha p} \fint_{B(x, r)} g^p d\mu = 0 \right\}.$$

Clearly, $\mu(X \setminus F) = 0$.

Now fix the ball $B = B(z, r) \subset X$. We will show that

$$\sup_{x, y \in B \cap F} |f(x) - f(y)| \leq c r^\alpha \left(\fint_{B(z, 2r)} g^p d\mu \right)^{\frac{1}{p}}.$$

Let $x, y \in B(z, r) \cap F$. By lemma 3 there exists a sequence of balls $B_i = B(x_i, c_0^i r) \subset B(x, r)$ such that

$$B_i \subset B(x, c_0^{i-1} r) \setminus B(x, c_0^i r).$$

It's easy to see that for any $i \in \mathbb{N}$ there exists a point $z_i \in B_i \cap F$ for which

$$g(z_i) \leq \left(\fint_{B_i} g^p d\mu \right)^{\frac{1}{p}}.$$

Hence we get the chain of inequalities

$$|f(x) - f(z_i)| \leq (d(x, z_i))^\alpha (g(x) + g(z_i)) \leq c r_i^\alpha \left(g(x) + \left(\fint_{B_i} g^p d\mu \right)^{\frac{1}{p}} \right).$$

The last inequality together with lemma 2 show that

$$\lim_{i \to \infty} f(z_i) = f(x).$$

Therefore, we have

$$|f(z_1) - f(x)| \leq \sum_{i=1}^\infty |f(z_i) - f(z_{i+1})| \leq \sum_{i=1}^\infty (d(z_i, z_{i+1}))^\alpha (g(z_i) + g(z_{i+1})) \leq$$

$$\leq c \sum_{i=1}^\infty r_i^\alpha \left(\left(\fint_{B_i} g^p d\mu \right)^{\frac{1}{p}} + \left(\fint_{B_{i+1}} g^p d\mu \right)^{\frac{1}{p}} \right) \leq c \sum_{i=1}^\infty r_i^\alpha \left(\fint_{B_i} g^p d\mu \right)^{\frac{1}{p}}.$$

Now, applying doubling condition and an elementary inequality

$$\sum_i a_i^{\frac{1}{p}} \leq \left(\sum_i a_i \right)^{\frac{1}{p}},$$

which holds for $0 < p \leq 1$, one can obtain

$$\left|f(z_1) - f(x)\right| \le c\sum_{i=1}^{\infty} r_i^{\alpha} \left(\oint_{B_i} g^p d\mu\right)^{\frac{1}{p}} = c\sum_{i=1}^{\infty} r_i^{\alpha} \left(\frac{\mu(B)}{\mu(B_i)} \frac{1}{\mu(B)} \int_{B_i} g^p d\mu\right)^{\frac{1}{p}} \le$$

$$\le cr^{\alpha} \sum_{i=1}^{\infty} \left(\frac{1}{\mu(B)} \int_{B_i} g^p d\mu\right)^{\frac{1}{p}} \le cr^{\alpha} \left(\frac{1}{\mu(B)} \sum_{i=1}^{\infty} \int_{B_i} g^p d\mu\right)^{\frac{1}{p}} \le cr^{\alpha} \left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}}. \tag{5}$$

Next inequality follows from the doubling condition and the inclusion $B_1 \subset B(z, 2r)$

$$g(z_1) \le \left(\oint_{B_1} g^p d\mu\right)^{\frac{1}{p}} = \left(\frac{1}{\mu(2B)} \frac{\mu(2B)}{\mu(B_1)} \int_{B_1} g^p d\mu\right)^{\frac{1}{p}} \le c\left(\oint_{2B} g^p d\mu\right)^{\frac{1}{p}}.$$

Similarly, one can find a point $w_1 \in B(y, r) \subset B(z, 2r)$ such that

$$\left|f(w_1) - f(x)\right| \le cr^{\alpha} \left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}} \tag{6}$$

and

$$g(w_1) \le c\left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}}.$$

It allows us to conclude that

$$g(z_1) + g(w_1) \le c\left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}}.$$

Finally, in order to get inequality (3), write down triangle inequality

$$\left|f(x) - f(y)\right| \le \left|f(x) - f(z_1)\right| + \left|f(z_1) - f(w_1)\right| + \left|f(w_1) - f(y)\right|,$$

and estimate first and third terms with help of (5) and (6) respectively. Then use the following inequality for the second term

$$\left|f(z_1) - f(w_1)\right| \le c\left(d(x, y)\right)^{\alpha}\left(g(z_1) + g(w_1)\right) \le cr^{\alpha} \left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}}.$$

Taking supremum we get

$$\sup_{x, y \in B \cap F} \left|f(x) - f(y)\right| \le cr^{\alpha} \left(\oint_{B(z,2r)} g^p d\mu\right)^{\frac{1}{p}}.$$

Now continuity follows from lemma 2. This completes the proof of theorem 2.

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