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НОВЫЕ ВЕРХНИЕ ГРАНИЦЫ ДЛЯ ФУНКЦИИ НЕЦЕНТРАЛЬНОГО ХИ-КВАДРАТ РАСПРЕДЕЛЕНИЯ

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Некоторые новые верхние границы для функции нецентрального хи-квадрат распределения выводятся из базовых симметрий плотности многомерного стандартного нормального закона: унитарной инвариантности, независимости компонент как в полярной, так и в декартовой системе координат. В сравнении с известными в литературе аналогами предложенные новые верхние оценки имеют простой аналитический вид: они представляют собой комбинации из экспонент, прямых и обратных тригонометрических функций, в том числе гиперболических, а также функции распределения одномерного стандартного нормального закона. Данные оценки могут быть полезны как в теории, так и в приложениях для доказательства неравенств, связанных с функцией нецентрального хи-квадрат распределения, и построения нижних оценок мощности хи-квадрат критерия Пирсона.

Ключевые слова: нецентральное хи-квадрат распределение; верхняя граница.

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NEW UPPER BOUNDS FOR NONCENTRAL CHI-SQUARE CDF

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Some new upper bounds for noncentral chi-square cumulative density function are derived from the basic symmetries of the multidimensional standard Gaussian distribution: unitary invariance, components independence in both polar and Cartesian coordinate systems. The proposed new bounds have analytically simple form compared to analogues available in the literature: they are based on combination of exponents, direct and inverse trigonometric functions, including hyperbolic ones, and the cdf of the one dimensional standard Gaussian law. These new bounds may be useful both in theory and in applications: for proving inequalities related to noncentral chi-square cumulative density function, and for bounding powers of Pearson's chi-squared tests.

Keywords: noncentral chi-square distribution; upper bound.

Introduction

Let $d \in \mathbb{N}$, $\mu = (\mu_i)_{i=1}^{d} \in \mathbb{R}^d$, $\lambda = \|\mu\|^2 = \sum_{i=1}^{d} \mu_i^2$. Then the cumulative distribution function (cdf) of the noncentral chi-square distribution with d degrees of freedom and noncentrality parameter λ is defined as follows:

$$f(x, d, \lambda) = \text{prob}\left\{\|\xi - \mu\|^2 \le x\right\}, \ x \ge 0. \tag{1}$$

Here $\xi \in \mathbb{R}^d$ is a standard normally distributed random d-vector. For the central chi-square cdf ($\lambda = 0$) we use brief notation $f(x, d) ::= f(x, d, 0)$.

The function (1) plays an important role in mathematical statistics. In particular, consider the classical problem of statistical hypothesis testing of null-hypothesis H_0: $L\{y_t\} = p = (p_i)_{i=1}^{K}$ (T observed i. i. d. random variables $\{y_t\}_{t=1}^{T}$, $y_t \in \{1, \ldots, K\}$, have their common mass probability function $p_i : \{1, \ldots, K\} \to \mathbb{R}_+$) against point alternative hypothesis H_1: $L\{y_t\} = q = (q_i)_{i=1}^{K}$. If the significance level $\alpha \in (0, 1)$ is fixed, and H_1 is contiguous to H_0, i. e. the following convergence holds:

$$T \sum_{i=1}^{K} \frac{(p_i - q_i)^2}{p_i} \to \lambda > 0$$

as $T \to \infty$, then the probability β of type 2 error of the standard Pearson's chi-squared test converges to $f(x, d, \lambda)$ with $d = K - 1$ and $x = F_{\chi_d^2}^{-1}(1 - \alpha)$ (quantile function for the central chi-square distribution with d degrees of freedom). Hence the upper bounds for (1) provide the lower bounds for asymptotic power of chi-squared test under contiguous alternatives.

The function (1) is well studied analytically, being closely related to the generalized Marcum functions [1; 2] and modified Bessel function of the first kind [3]. Various upper and lower bounds for (1) are also available in the literature [1; 4]. Analytical expressions for these bounds, however, are as complex as the ones for $f(x, d, \lambda)$, being based on transcendental functions like modified Bessel function [1] or the moments of truncated normal distribution [4]. We present here some new upper bounds for (1). These bounds are of a relatively simple analytical form and may be useful both in theory (proving inequalities related to (1)) and in applications (bounding powers of chi-squared tests).

Upper bounds for noncentral chi-square cdf

Since the value (1) equals a standard Gaussian measure of a ball $B_{\mu, \sqrt{x}}$ of radius $\sqrt{x}$ with center μ, our idea is to construct upper bounds of the form

$$f(x, d, \lambda) \le \text{prob}\{\xi \in A\}, \ B_{\mu, \sqrt{x}} \subset A \subset \mathbb{R}^d. \tag{2}$$

Let $\Pi_1, \Pi_2 \subset \mathbb{R}^d$ be orthogonally complemented subspaces. Minkowski sums $A_i = B_{\mu, \sqrt{x}} + \Pi_i$ are cylindric sets containing $B_{\mu, \sqrt{x}}$. Due to properties of standard normal distribution, the events $\xi \in A_i$ are independent and $\operatorname{prob}\{\xi \in A_i\} = f(x, d_i, \lambda_i)$, where $d_i = \dim\Pi_i$ and λ_i is a squared norm of an orthogonal projection of μ onto Π_i, $i = 1, 2$. The set $A = A_1 \cap A_2$ in (2) leads to the following upper bound.

Lemma 1. *Let $d = d_1 + d_2$, $\lambda = \lambda_1 + \lambda_2$, $\lambda_i \geq 0$, $d_i \in \mathbb{N}$, $i = 1, 2$. Then the following inequality holds:*

$$f(x, d, \lambda) \leq f(x, d_1, \lambda_1) f(x, d_2, \lambda_2). \tag{3}$$

Since $f(x, 1, \lambda) = \Phi\Big|_{\sqrt{\lambda} - \sqrt{x}}^{\sqrt{\lambda} + \sqrt{x}}$, where $\Phi(\cdot)$ is the standard Gaussian cdf, we get from (3):

$$f(x, d, \lambda) \leq f(x, d-1)\Phi\Big|_{\sqrt{\lambda} - \sqrt{x}}^{\sqrt{\lambda} + \sqrt{x}}. \tag{4}$$

Repeated application of (3) also gives the following bounds:

$$f(x, d, \lambda) \leq \left(\Phi\Big|_{\sqrt{\lambda/d} - \sqrt{x}}^{\sqrt{\lambda/d} + \sqrt{x}}\right)^d, \tag{5}$$

$$f(x, d, \lambda) \leq \left(\Phi\Big|_{-\sqrt{x}}^{\sqrt{x}}\right)^{d-1} \Phi\Big|_{\sqrt{\lambda} - \sqrt{x}}^{\sqrt{\lambda} + \sqrt{x}}. \tag{6}$$

Another way to construct covering set A in (2) is based on unitary invariance of standard normal distribution. Namely, let us assume $d \geq 2$, $x \leq \lambda$, and define

$$A_1 = \left\{w \in \mathbb{R}^d : \left|\|w\| - \sqrt{\lambda}\right| \leq x\right\},\ A_2 = \left\{cw : c \geq 0,\ w \in B_{\mu, \sqrt{x}}\right\}.$$

According to mentioned unitary invariance, the events $\xi \in A_i$ are independent as well. It is easy to see that $\operatorname{prob}\{\xi \in A_1\} = f(\cdot, d)\Big|_{(\sqrt{\lambda} - \sqrt{x})^2}^{(\sqrt{\lambda} + \sqrt{x})^2}$, while A_2 is a cone and $\operatorname{prob}\{\xi \in A_2\}$ equals normalized Lebesgue measure of a spherical ball $A_2 \cap S^{d-1}$ of radius $\arcsin\sqrt{\frac{x}{\lambda}}$ (in spherical metric, S^{d-1} is the unit sphere in $\mathbb{R}^d$). Hence we get the following lemma.

Lemma 2. *The following inequality holds for $d \geq 2$, $x \leq \lambda$:*

$$f(x, d, \lambda) \leq f(\cdot, d)\Big|_{(\sqrt{\lambda} - \sqrt{x})^2}^{(\sqrt{\lambda} + \sqrt{x})^2} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}} \int_0^{\arcsin\sqrt{\frac{x}{\lambda}}} (\sin\rho)^{d-2} d\rho, \tag{7}$$

where $\Gamma(\cdot)$ is Gamma function.

Using the inequalities $\operatorname{prob}\{\xi \in A_2\} \leq \frac{1}{2}$, $\frac{\Gamma\left(z + \frac{1}{2}\right)}{\Gamma(z)} \leq \sqrt{z}$, $z > 0$, and

$$\int_0^{\rho_*} (\sin\rho)^{d-2} d\rho \leq \int_0^{\rho_*} (\sin\rho)^{d-2} \frac{d\sin\rho}{\cos\rho_*} = \frac{(\sin\rho_*)^{d-1}}{(d-1)\cos\rho_*},\ \forall\rho_* \in \left(0, \frac{\pi}{2}\right),$$

we obtain a weakened version of (7) having more explicit form:

$$f(x, d, \lambda) \leq f(\cdot, d)\Big|_{(\sqrt{\lambda} - \sqrt{x})^2}^{(\sqrt{\lambda} + \sqrt{x})^2} \min\left\{\frac{1}{2}, \sqrt{\frac{\left(\frac{x}{\lambda}\right)^{d-1}}{2\pi(d-1)\left(1 - \frac{x}{\lambda}\right)}}\right\},\ d \geq 2,\ x \leq \lambda. \tag{8}$$

For even d the bound (7) has completely explicit form since central chi-square pdf is integrable.

Corollary. *The following inequalities hold for* $x \le \lambda$:

$$f(x, 2, \lambda) \le \frac{2}{\pi} e^{-\frac{1}{2}(\lambda + x)} \sinh\sqrt{\lambda x} \arcsin\sqrt{\frac{x}{\lambda}}, \tag{9}$$

$$f(x, 4, \lambda) \le \frac{2}{\pi} e^{-\frac{1}{2}(\lambda + x)} \left(\left(1 + \frac{\lambda + x}{2}\right) \sinh\sqrt{\lambda x} - \sqrt{\lambda x}\cosh\sqrt{\lambda x} \right) \times$$
$$\times \left(\arcsin\sqrt{\frac{x}{\lambda}} - \lambda^{-1}\sqrt{x(\lambda - x)} \right). \tag{10}$$

Combining (3) with (9), we get the following bounds for even $d = 2k$ and $x \le \dfrac{\lambda}{k}$:

$$f(x, 2k, \lambda) \le e^{-\frac{1}{2}(\lambda + kx)} \left(\frac{2}{\pi} \sinh\sqrt{\frac{\lambda x}{k}} \arcsin\sqrt{\frac{kx}{\lambda}} \right)^k, \tag{11}$$

$$f(x, 2k, \lambda) \le \frac{2}{\pi} e^{-\frac{1}{2}(\lambda + kx)} \sinh^{k-1}(x) \sinh\sqrt{\lambda_* x} \arcsin\sqrt{\frac{x}{\lambda_*}}, \tag{12}$$

where $\lambda_* = \lambda - (k - 1)x$. The bounds similar to (11), (12) can be obtained from (10) for $d = 4k$.

Computer experiments

The four plots on the fig. 1 illustrate the upper bounds for (1) proposed in the paper. On the plots on fig. 1, *a*, *b*, *d*, we see that the corresponding upper bounds are strictly ordered for the chosen d and λ. This observation allows us to formulate the conjecture that the upper bounds for (1) are ordered as follows: «(6) ≤ (5)» for any $x \ge 0$, $d \in \mathbb{N}$, $\lambda \ge 0$; «(11) ≤ (12)» for any $0 \le x \le \dfrac{\lambda}{k}$, $d = 2k$ (even), $\lambda \ge 0$; «(10) ≤ (6) ≤ (11)» for any $0 \le x \le \dfrac{\lambda}{2}$, $d = 4$, $\lambda \ge 0$. The inequality «(4) ≤ (6)» is not listed, because it obviously follows from (5). The plot fig. 1, *c*, allows to conjecture that for $d \ge 2$ the upper bound (8) is better than (4) for small $x \le x_*$ up to some $x_* \le \lambda_*$, and vice versa for $x_* \le x \le \lambda_*$.

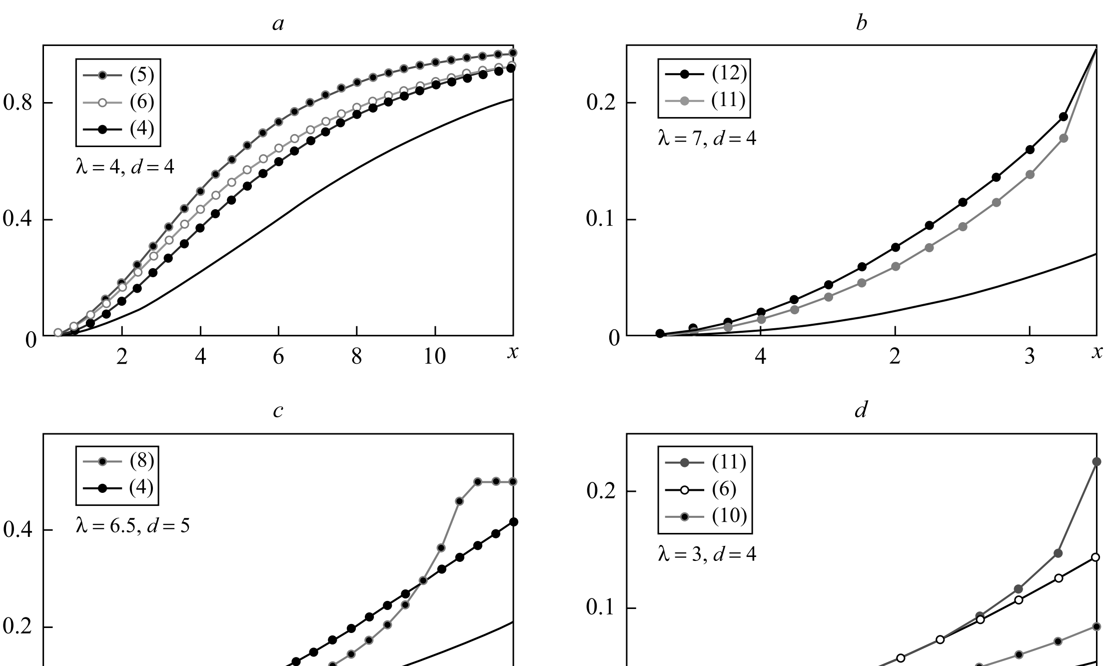

Fig. 1. Noncentral chi-square cdf (1) (lower black lines) and its upper bounds (upperbroken lines)

The plot on the fig. 2 illustrates dependence of the type 2 error β of chi-square test on data volume T under alternative H_1 contiguous to H_0:

$$T\sum_{i=1}^{K}\frac{\left(p_i-q_i\right)^2}{p_i}\to\lambda=7,$$

where $K=5$; $p=\left(p_i\right)=\left(0.05,\,0.1,\,0.15,\,0.3,\,0.4\right)$. The threshold of chi-square test $x=6$ corresponds to the significance level $\alpha=1-F_{\chi^2_{K-1}}\left(x\right)\approx0.2$. The upper level $\beta_{\max}\approx0.265$ on the plot corresponds to the upper bound (10) for $x=6$, $\lambda=7$.

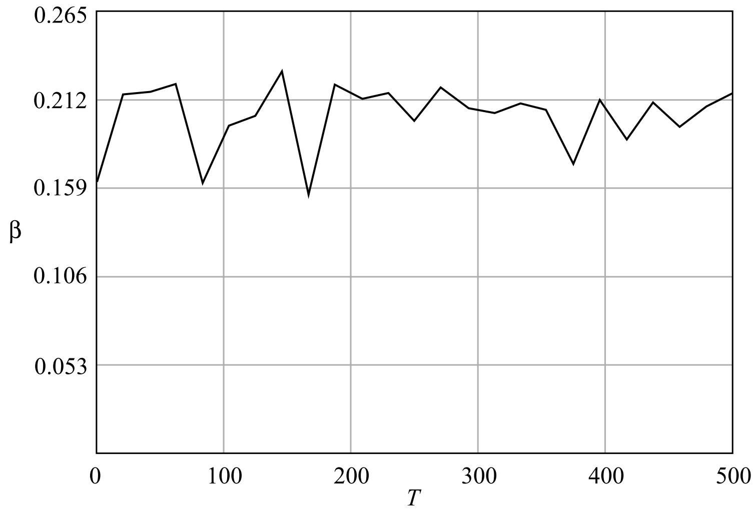

Fig. 2. Chi-square test type 2 error β against data volume T

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