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ОБ ОДНОМ РАЦИОНАЛЬНОМ ИНТЕГРАЛЬНОМ ОПЕРАТОРЕ ТИПА ФУРЬЕ – ЧЕБЫШЁВА И АППРОКСИМАЦИИ ФУНКЦИЙ МАРКОВА

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Авторами статьи ставится цель построить рациональный интегральный оператор типа Фурье на основании системы рациональных функций Чебышёва – Маркова и исследовать его аппроксимационные свойства на классах функций Маркова. Вводится интегральный оператор типа Фурье – Чебышёва на основании рациональных функций Чебышёва – Маркова, представляющий собой рациональную функцию порядка не выше n, и изучаются приближения

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функций Маркова. Получены интегральное представление и равномерная оценка приближений. В случае когда мера μ удовлетворяет следующим условиям: $\operatorname{supp}\mu = [1, a]$, $a > 1$, $d\mu(t) = \varphi(t)dt$ и $\varphi(t) \asymp (t-1)^{\alpha}$ на $[1, a]$, установлены оценки поточечных и равномерных приближений и асимптотическое выражение при $n \to \infty$ мажоранты равномерных приближений. При фиксированном количестве геометрически различных полюсов в расширенной комплексной плоскости найдены оптимальные значения параметров, обеспечивающие наибольшую скорость убывания этой мажоранты, а также асимптотически точные оценки наилучших равномерных приближений этим методом при четном количестве геометрически различных полюсов аппроксимирующей функции. Приведены асимптотические оценки приближений некоторых элементарных функций, представимых функциями Маркова.

Ключевые слова: функция Маркова; рациональный интегральный оператор типа Фурье; рациональная функция Чебышёва – Маркова; мажоранта равномерных приближений; асимптотическая оценка; наилучшее приближение; точная константа.

ON ONE RATIONAL INTEGRAL OPERATOR OF FOURIER – CHEBYSHEV TYPE AND APPROXIMATION OF MARKOV FUNCTIONS

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The purpose of this paper is to construct an integral rational Fourier operator based on the system of Chebyshev – Markov rational functions and to study its approximation properties on classes of Markov functions. In the introduction the main results of well-known works on approximations of Markov functions are present. Rational approximation of such functions is a well-known classical problem. It was studied by A. A. Gonchar, T. Ganelius, J.-E. Andersson, A. A. Pekarskii, G. Stahl and other authors. In the main part an integral operator of the Fourier – Chebyshev type with respect to the rational Chebyshev – Markov functions, which is a rational function of order no higher than n is introduced, and approximation of Markov functions is studied. If the measure satisfies the following conditions: $\operatorname{supp}\mu = [1, a]$, $a > 1$, $d\mu(t) = \varphi(t)dt$ and $\varphi(t) \asymp (t-1)^{\alpha}$ on $[1, a]$, the estimates of pointwise and uniform approximation and the asymptotic expression of the majorant of uniform approximation are established. In the case of a fixed number of geometrically distinct poles in the extended complex plane, values of optimal parameters that provide the highest rate of decreasing of this majorant are found, as well as asymptotically accurate estimates of the best uniform approximation by this method in the case of an even number of geometrically distinct poles of the approximating function. In the final part we present asymptotic estimates of approximation of some elementary functions, which can be presented by Markov functions.

Keywords: Markov function; integral rational operator of Fourier type; Chebyshev – Markov rational function; majorant of uniform approximation; asymptotic estimate; best approximation; exact constant.

Introduction

Let μ be positive Borel measure with a compact support $F = \operatorname{supp}\mu \subset \mathbb{R}$. Cauchy transform of the measure μ

$$\hat{\mu}(z) = \int_F \frac{d\mu(t)}{t - z}, \ z \in \mathbb{C},$$

is called a Markov function. Approximation of Markov functions is well-known classical problem in the theory of rational approximation of analytic functions. One of the first works, devoted to the study of rational approximation of Markov functions, is an article by A. A. Gonchar [1]. T. Ganelius [2] applied some well-known results from the theory of orthogonal polynomials and interpolation theory to the problems of rational approximation of Markov functions. Subsequently, quite important results related to the rational approximation of Markov functions belong to J.-E. Andersson [3], A. A. Pekarskii [4], D. Braess [5]. This issue was further developed by many authors (see, for example, [6–8]).

To date, methods based on Fourier series with respect to orthogonal systems of rational functions are widely used. These methods were also applied in studies devoted to rational approximation of Markov functions. A. A. Pekarskii and Y. A. Rouba [9] investigated rational approximation of Markov functions by partial sums of Fourier series with respect to the systems of functions in the unit circle, introduced by S. Takenaka [10] and F. Malmquist [11], and systems of functions on the segment $[-1, 1]$, introduced by M. M. Dzhrbashyan and A. A. Kitbalyan [12].

K. N. Lungu [13; 14] studied approximation of continuous functions on a segment by rational functions of degree no higher than n and with no more than q $(0 \le q < n)$ geometrically distinct poles in a finite (extended) complex plane and obtained a number of results in this direction. Based on the integral representation of deviation of partial sums of Fourier series from Markov function obtained in [9], Y. A. Rouba and Y. G. Mikulich [15] found asymptotic estimates of uniform approximation, when approximating function has fixed number of geometrically distinct poles. In other words, they solved the problem of K. N. Lungu, extended to the class of Markov functions and the partial sums of rational Fourier series were used as a method of approximation.

The main purpose of this work is to study the approximation of Markov functions by an integral operator of the Fourier type based on the system of Chebyshev – Markov rational functions. The integral representation of deviation of this operator from Markov function is established. In the case when the measure μ satisfies the conditions: $\operatorname{supp}\mu = [1, a]$, $a > 1$, $d\mu(t) = \varphi(t)dt$ and $\varphi(t) \asymp (t-1)^{\alpha}$ on $[1, a]$, estimates of pointwise and uniform approximations are found. These estimates are exact when multiplicity of poles of the approximating function is even. In this case an asymptotic expression for the majorant of uniform approximations when $n \to \infty$ is also established.

Further in the paper we consider approximation of the classes of Markov functions by means of rational functions with a fixed number of geometrically distinct poles of even multiplicity. In this case, using the Laplace method [16; 17], we establish the asymptotic behaviour of the majorant and prove the order of uniform approximation. It should be noted that similar results for the approximation of the function $|x|^s$ by integral operator of Fejer type were obtained in [18].

Also the examples of approximation of some elementary functions, which can be represented as Markov functions, are considered.

System of Chebyshev – Markov rational fractions

Let the numbers $\{a_k\}_{k=1}^{n}$ be real and $|a_k| < 1$ or be paired by complex conjugation. Consider the Chebyshev – Markov rational fraction

$$M_n(x) = \cos \sum_{k=1}^{n} \arccos \frac{x + a_k}{1 + a_k x}, \quad x \in [-1, 1]. \tag{1}$$

Note, that if all the numbers $a_k = 0$, $k = 1, 2, \ldots, n$, then the functions $M_n(x)$ degenerate to the classical Chebyshev polynomials of the first kind.

Now we consider some properties of functions (1).

Lemma 1. *The Chebyshev – Markov rational fraction (1) is as follows*

$$M_n(x) = \frac{p_n(x)}{\prod_{k=1}^{n}(1 + a_k x)},$$

where $p_n(x)$ *is an algebraic polynomial of degree n with coefficients depending on* a_k, $k = 1, 2, \ldots, n$.

P r o o f. Using the method, applied in [19], we immediately obtain that

$$M_n(x) = \frac{1}{2}\left[\prod_{k=1}^{n} \frac{x + a_k + i\sqrt{1-x^2}\sqrt{1-a_k^2}}{1 + a_k x} + \prod_{k=1}^{n} \frac{x + a_k - i\sqrt{1-x^2}\sqrt{1-a_k^2}}{1 + a_k x}\right]. \tag{2}$$

The result of lemma 1 follows from equality (2) and the conditions imposed on the parameters $\{a_k\}_{k=1}^{n}$.

Lemma 2. *The following representation holds*

$$M_n(x) = \frac{1}{2}\left[\prod_{k=1}^{n} \frac{\xi + \alpha_k}{1 + \alpha_k \xi} + \prod_{k=1}^{n} \frac{1 + \alpha_k \xi}{\xi + \alpha_k}\right], \tag{3}$$

where $\xi = e^{iu}$, $x = \cos u$, $\alpha_k = \dfrac{a_k}{1 + \sqrt{1 - a_k^2}}$, $|\alpha_k| < 1$, $k = 1, 2, \ldots, n$.

P r o o f. Let $x = \cos u$ in (2). Then

$$M_n(x) = \frac{1}{2}\left[\prod_{k=1}^{n} \frac{\cos u + a_k + i\sin u\sqrt{1-a_k^2}}{1+a_k\cos u} + \prod_{k=1}^{n} \frac{\cos u + a_k - i\sin u\sqrt{1-a_k^2}}{1+a_k\cos u}\right].$$

Substituting $\xi = \mathrm{e}^{iu}$ and using Euler's formula after some necessary transformations we get the representation (3). Lemma 2 is proved.

Rational integral operator of Fourier – Chebyshev type and its Dirichlet kernel

Let f be an absolutely integrable function with respect to the weight $\left(1-x^2\right)^{-1/2}$ on the segment $[-1, 1]$. Consider an integral operator $s_n : f \to s_n(f, x)$, which is defined by the formula

$$s_n(f, x) = \frac{1}{2\pi\gamma_n}\int_{-1}^{+1} f(t)\frac{M_{n+1}(t)M_n(x) - M_{n+1}(x)M_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}}, \quad n = 0, 1, \ldots, \tag{4}$$

where $M_n(x)$ is a Chebyshev – Markov rational fraction (1). Require this operator to be exact for the constants, it means $s_n(1, x) \equiv 1$. We find γ_n accordingly.

Lemma 3. *The following equality holds*

$$\gamma_n = \frac{1-\alpha_{n+1}^2}{2\left(1+2\alpha_{n+1}\cos u + \alpha_{n+1}^2\right)}, \quad x = \cos u, \ \left|\alpha_{n+1}\right| < 1, \ n = 1, 2, \ldots. \tag{5}$$

P r o o f. It is clear, that

$$\gamma_n = \frac{1}{2\pi}\int_{-1}^{+1} \frac{M_{n+1}(t)M_n(x) - M_{n+1}(x)M_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}}, \quad n = 0, 1, \ldots.$$

Let $x = \cos u$. In the last integral we substitute $t = \cos v$. Then

$$\gamma_n = \frac{1}{8\pi}\int_0^{\pi} \frac{\left(\omega_{n+1}(\zeta) + \overline{\omega_{n+1}(\zeta)}\right)\left(\omega_n(\xi) + \overline{\omega_n(\xi)}\right) - \left(\omega_{n+1}(\xi) + \overline{\omega_{n+1}(\xi)}\right)\left(\omega_n(\zeta) + \overline{\omega_n(\zeta)}\right)}{\cos v - \cos u}dv,$$

where

$$\omega_n(z) = \prod_{k=1}^{n}\frac{z+\alpha_k}{1+\alpha_k z}, \quad \zeta = \mathrm{e}^{iv}, \ \xi = \mathrm{e}^{iu}. \tag{6}$$

After some transformations, we obtain

$$\gamma_n = \frac{1-\alpha_{n+1}^2}{8\pi}\int_{-\pi}^{+\pi}\left[\frac{\xi\zeta - 1}{\left(1+\alpha_{n+1}\zeta\right)\left(\xi+\alpha_{n+1}\right)}\frac{\omega_n(\zeta)}{\omega_n(\xi)} - \frac{\xi\zeta - 1}{\left(1+\alpha_{n+1}\xi\right)\left(\zeta+\alpha_{n+1}\right)}\frac{\omega_n(\xi)}{\omega_n(\zeta)}\right]\frac{dv}{\cos v - \cos u}. \tag{7}$$

Note that in this integral (7) the point $\zeta = \xi$ is not singular, since it is also zero of the numerator of the integrand. Assume here $\xi = \rho\mathrm{e}^{iu}$, $\rho < 1$, and again we apply substitution $\zeta = \mathrm{e}^{iv}$. Then

$$\gamma_n = \frac{\left(1-\alpha_{n+1}^2\right)\xi}{4\pi i}\left[\frac{1}{\omega_n(\xi)\left(\xi+\alpha_{n+1}\right)}\oint_{|\zeta|=1}\frac{\omega_n(\zeta)d\zeta}{\left(1+\alpha_{n+1}\zeta\right)(\zeta-\xi)} - \frac{\omega_n(\xi)}{\left(1+\alpha_{n+1}\xi\right)}\oint_{|\zeta|=1}\frac{d\zeta}{\omega_n(\zeta)\left(\zeta+\alpha_{n+1}\right)(\zeta-\xi)}\right].$$

The integrand of the second integral in the square brackets has no isolated singular points outside the unit disk and has infinity as zero of the second order at least. Therefore, this integral is equal to zero.

The integrand of the first integral has the only singular point inside the unit disk $\zeta = \xi$ as a simple pole. Applying the Cauchy residue theorem and passing to the limit as $\rho \to 1$, we obtain the equality (5). This concludes the proof of lemma 3.

Remark 1. If in (4) $a_k = 0$, $k = 1, 2, \ldots, n$, then the operator $s_n(\cdot, x)$ degenerates to the partial sum of Fourier series with respect to the Chebyshev polynomials of the first kind [20].

Corollary 1. *For the integral operator (4) the following representation holds*

$$s_n(f, x) = \int_{-\pi}^{+\pi} f(\cos v)\mathfrak{D}_n(u, v)dv, \tag{8}$$

where

$$\mathfrak{D}_n(u, v) = \frac{1}{2\pi}\left[\frac{1+\alpha_{n+1}\xi}{1+\alpha_{n+1}\zeta}\frac{\omega_n(\zeta)}{\omega_n(\xi)} - \frac{\xi+\alpha_{n+1}}{\zeta+\alpha_{n+1}}\frac{\omega_n(\xi)}{\omega_n(\zeta)}\right]\frac{\zeta}{\zeta-\xi},\ \zeta = e^{iv},\ \xi = e^{iu},\ x = \cos u, \tag{9}$$

$\omega_n(z)$ *is defined by (6).*

P r o o f. From lemma 3 and formula (4) we get

$$s_n(f, x) = \frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{\pi\left(1-\alpha_{n+1}^2\right)}\int_{-1}^{+1} f(t)\frac{M_{n+1}(t)M_n(x) - M_{n+1}(x)M_n(t)}{t - x}\frac{dt}{\sqrt{1-t^2}},$$

where $M_n(x)$ is a Chebyshev – Markov rational fraction (1). Again we apply substitution $t = \cos v$, assuming $x = \cos u$. Then

$$s_n(f, x) = \frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{\pi\left(1-\alpha_{n+1}^2\right)}\int_{0}^{\pi} f(\cos v)\frac{M_{n+1}(\cos v)M_n(\cos u) - M_{n+1}(\cos u)M_n(\cos v)}{\cos v - \cos u}dv.$$

Using representation 3 and arguments of lemma 3, we get formula (8). Corollary 1 is proved.

Theorem 1. *For the integral operator (4) the following representation holds*

$$s_n(f, x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\cos v)\sqrt{\frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 + 2\alpha_{n+1}\cos v + \alpha_{n+1}^2}}\frac{\sin\lambda_n(v, u)}{\sin\frac{v-u}{2}}dv,\ x = \cos u, \tag{10}$$

where

$$\lambda_n(v, u) = \int_u^v \lambda_n(y)dy,\ \lambda_n(y) = \frac{1-\alpha_{n+1}^2}{2\left(1 + 2\alpha_{n+1}\cos y + \alpha_{n+1}^2\right)} + \sum_{k=1}^{n}\frac{1-\alpha_k^2}{1 + 2\alpha_k\cos y + \alpha_k^2},\ |\alpha_k| < 1.$$

P r o o f. Let us consider the kernel (9). We have

$$\mathfrak{D}_n(u, v) = \frac{1}{4\pi i}\left[\sqrt{\frac{\zeta}{\xi}}\frac{1+\alpha_{n+1}\xi}{1+\alpha_{n+1}\zeta}\frac{\omega_n(\zeta)}{\omega_n(\xi)} - \sqrt{\frac{\zeta}{\xi}}\frac{\xi+\alpha_{n+1}}{\zeta+\alpha_{n+1}}\frac{\omega_n(\xi)}{\omega_n(\zeta)}\right]\left(\sin\frac{v-u}{2}\right)^{-1}, \tag{11}$$

where $\zeta = e^{iv}$, $\xi = e^{iu}$. The expression in square brackets is a difference of two complex conjugate terms. Let us transform it. Applying the same considerations as in [21], we get

$$\frac{\omega_n(\zeta)}{\omega_n(\xi)} = \exp\left[i\int_u^v\sum_{k=1}^{n}\frac{1-\alpha_k^2}{1 + 2\alpha_k\cos y + \alpha_k^2}dy\right],\ |\alpha_k| < 1,\ k = 1, \ldots, n.$$

Similarly,

$$\frac{1+\alpha_{n+1}\xi}{1+\alpha_{n+1}\zeta} = \sqrt{\frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 + 2\alpha_{n+1}\cos v + \alpha_{n+1}^2}}\exp\left[-i\int_u^v\frac{\alpha_{n+1}\cos y + \alpha_{n+1}^2}{1 + 2\alpha_{n+1}\cos y + \alpha_{n+1}^2}dy\right].$$

Plugging the last expression in (11) and noticing, that $\sqrt{\frac{\zeta}{\xi}} = \exp\left[\frac{i(v-u)}{2}\right]$, we obtain formula (10). Theorem 1 is proved.

Remark 2. If $\alpha_{n+1} = 0$, then the expression (10) defines an integral operator based on Chebyshev – Markov rational functions, studied in [22].

Remark 3. If in (10) all the parameters $\alpha_k = 0$, $k = 1, 2, \ldots, n+1$, then

$$s_n(f, x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\cos v)\frac{\sin\left(n+\frac{1}{2}\right)(v-u)}{\sin\frac{v-u}{2}}dv.$$

In other words, in this case $s_n(f, x)$ is a partial sum of polynomial Fourier – Chebyshev series of the function f.

Approximation of Markov functions

Now we will investigate approximation of Markov functions $\hat{\mu}(x)$ by rational integral operator (4) in uniform metric on the segment $[-1, 1]$.

Let

$$\varepsilon_n(x, A) = \hat{\mu}(x) - s_n\left(\hat{\mu}(x), x\right), \ x \in [-1, 1], \ \varepsilon_n(A) = \left\| \hat{\mu}(x) - s_n\left(\hat{\mu}(x), x\right)\right\|_{C[-1,1]}, \ n \in \mathbb{N},$$

where $A = \left\{\alpha : \alpha = \left(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\right), \ |\alpha_k| < 1\right\}$ is a set of $(n+1)$-dimensional vectors. Also we assume, that $\operatorname{supp}\mu \subset [1, +\infty)$ and

$$\int \frac{d\mu(t)}{t-1} < \infty. \tag{12}$$

Theorem 2. *Let measure* μ *satisfy the condition (12) and measure* ν *is defined by formula*

$$d\nu(y) = \frac{4y^2}{1-y^2}d\mu(\varphi(y)), \ y \in (0, 1], \tag{13}$$

where

$$\varphi(y) = \frac{1}{2}\left(y + \frac{1}{y}\right).$$

Then for the approximation the following equality holds for $x \in [-1, 1]$

$$\varepsilon_n(x, A) = \int \sqrt{\frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 - 2y\cos u + y^2}}\cos\psi_n(u, y)\frac{\omega_n(y)}{1+\alpha_{n+1}y}d\nu(y), \ x = \cos u, \tag{14}$$

where

$$\psi_n(u, y) = \arg\frac{(1-\xi y)\left(1+\alpha_{n+1}\xi\right)}{\xi\omega_n(\xi)}, \ \omega_n(\xi) = \prod_{k=1}^{n}\frac{\xi + \alpha_k}{1+\alpha_k\xi}, \ \xi = e^{iu}. \tag{15}$$

P r o o f. Since the operator $s_n(\cdot, x)$ is exact for constant, using representation (8) for the approximation $\varepsilon_n(x, A)$ we find, that

$$\varepsilon_n(x, A) = \int \frac{1}{t - \cos u}I_n(u, t)d\mu(t), \ x = \cos u, \tag{16}$$

where

$$I_n(u, t) = \int_{-\pi}^{+\pi}\frac{\cos u - \cos v}{t - \cos v}\mathfrak{D}_n(u, v)dv,$$

and $\mathfrak{D}_n(u, v)$ is defined by (9). Assuming $\xi = e^{iu}$, in the last integral we substitute $\zeta = e^{iv}$. Then

$$I_n(u, t) = \frac{1}{2\pi i\xi}\oint_{|\zeta|=1}\frac{\xi\zeta - 1}{(\zeta - y)\left(\zeta - \frac{1}{y}\right)}\left[\frac{1+\alpha_{n+1}\xi}{1+\alpha_{n+1}\zeta}\frac{\omega_n(\zeta)}{\omega_n(\xi)} - \frac{\xi + \alpha_{n+1}}{\zeta + \alpha_{n+1}}\frac{\omega_n(\xi)}{\omega_n(\zeta)}\right]d\zeta,$$

where $y = y(t) = t - \sqrt{t^2 - 1},\ y \in (0, 1]$. Splitting this integral into two ones, we get

$$I_n(u, t) = \frac{1}{2\pi i\xi}\Big[\big(1 + \alpha_{n+1}\xi\big)\overline{\omega_n(\xi)}I_1(u, t) - \big(\xi + \alpha_{n+1}\big)\omega_n(\xi)I_2(u, t)\Big], \tag{17}$$

where

$$I_1(u, t) = \oint\limits_{|\zeta|=1} \frac{\xi\zeta - 1}{(\zeta - y)\left(\zeta - \frac{1}{y}\right)}\frac{\omega_n(\zeta)}{1 + \alpha_{n+1}\zeta}d\zeta,\ I_2(u, t) = \oint\limits_{|\zeta|=1} \frac{\xi\zeta - 1}{(\zeta - y)\left(\zeta - \frac{1}{y}\right)}\frac{d\zeta}{\omega_n(\zeta)\left(\zeta + \alpha_{n+1}\right)}.$$

Consider each of these integrals separately. Inside the unit circle the integrand of $I_1(u, t)$ has the only singular point $\zeta = y$ as a simple pole. Applying the Cauchy residue theorem, we obtain

$$I_1(u, t) = 2\pi i\frac{\xi y - 1}{y - \frac{1}{y}}\frac{\omega_n(y)}{1 + \alpha_{n+1}y},\ \xi = \mathrm{e}^{iu},\ y \in (0, 1]. \tag{18}$$

Outside the unit circle the integrand of $I_2(u, t)$ has the $\zeta = \frac{1}{y}$ as a simple pole and infinity as zero of the second order at least. Therefore,

$$I_2(u, t) = -2\pi i\frac{\xi - y}{\frac{1}{y} - y}\frac{\omega_n(y)}{1 + \alpha_{n+1}y},\ \xi = \mathrm{e}^{iu},\ y \in (0, 1]. \tag{19}$$

Plugging (18) and (19) into (17), we get

$$I_n(u, t) = \left[\frac{1 - \xi y}{\xi}\big(1 + \alpha_{n+1}\xi\big)\overline{\omega_n(\xi)} + \frac{\xi - y}{\xi}\big(\xi + \alpha_{n+1}\big)\omega_n(\xi)\right]\frac{\omega_n(y)}{\left(\frac{1}{y} - y\right)\left(1 + \alpha_{n+1}y\right)}.$$

Note, that since the terms in square brackets are complex conjugate, their sum is real. Besides $1 - \xi y = \sqrt{1 - 2y\cos u + y^2}\,\mathrm{e}^{i\arg(1 - \xi y)}$, and $1 + \alpha_{n+1}\xi = \sqrt{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}\,\mathrm{e}^{i\arg\left(1 + \alpha_{n+1}\xi\right)}$. So, we have

$$I_n(u, t) = \frac{2\sqrt{1 - 2y\cos u + y^2}\sqrt{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}\cos\psi_n^*(u, t)\omega_n(y)}{\left(\frac{1}{y} - y\right)\left(1 + \alpha_{n+1}y\right)}, \tag{20}$$

where

$$\psi_n^*(u, t) = \arg\frac{(1 - \xi y)\left(1 + \alpha_{n+1}\xi\right)}{\xi\omega_n(\xi)},\ \xi = \mathrm{e}^{iu},\ y = t - \sqrt{t^2 - 1}.$$

Plugging (20) into (16), we obtain

$$\varepsilon_n(x, A) = 2\int \frac{\sqrt{1 - 2y\cos u + y^2}\sqrt{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}\cos\psi_n^*(u, t)\omega_n(y)}{(t - \cos u)\left(\frac{1}{y} - y\right)\left(1 + \alpha_{n+1}y\right)}d\mu(t),\ x = \cos u.$$

In the last integral we do substitution $y = t - \sqrt{t^2 - 1}$. Then

$$\varepsilon_n(x, A) = \int\sqrt{\frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 - 2y\cos u + y^2}}\cos\psi_n(u, y)\frac{\omega_n(y)}{1 + \alpha_{n+1}y}\frac{4y^2}{1 - y^2}d\mu\big(\varphi(y)\big),\ x = \cos u,$$

where $\varphi(y)$ is Zhukovsky function, which is defined in (13), $\psi_n(u, y) = \psi_n^*\big(u, \varphi(y)\big)$. Finally, to get the representation (14), we need to use the formula (13). Theorem 2 is proved.

Corollary 2. *Under the conditions of theorem 2 uniformly for* $x \in [-1, 1]$, $x = \cos u$, *the following inequality holds*

$$\left|\varepsilon_n(x, A)\right| \le \varepsilon_n(A) \le \left\| \int\limits_{\operatorname{supp}\nu} \sqrt{\frac{1 + 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 - 2y\cos u + y^2}} \left|\omega_n(y)\right| \frac{d\nu(y)}{1 + \alpha_{n+1} y} \right\|_{C[-1, 1]},$$

where $\varepsilon_n(A)$ *is a uniform approximation of Markov functions by the integral operator (4).*

Estimates of pointwise and uniform approximation of Markov functions

In this section we are going to study the case, when the derivative of a measure $\mu(t)$ is weakly equivalent to some power function. Some previous results in this direction can be found, for example in [2; 3].

Theorem 3. *Let* $\operatorname{supp}\nu \in [d, 1]$, $0 \le d < 1$, $d\mu(t) = \varphi(t)dt$, *and* $\varphi(t) \asymp (t-1)^\gamma$. *Provided the conditions of theorem 2 are satisfied for the approximation of the function* $\hat{\mu}(x)$ *on the segment* $[-1, 1]$ *the following inequalities hold:*

1) for the pointwise approximation

$$\left|\varepsilon_n(x, A)\right| \le \frac{1}{2^{\gamma-1}} \int\limits_d^1 \sqrt{\frac{1 - 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 - 2y\cos u + y^2}} \frac{(1-y)^{2\gamma}}{y^\gamma} \frac{\left|\chi_n(y)\right|}{1 - \alpha_{n+1} y} dy, \quad x = \cos u; \tag{21}$$

2) for the uniform approximation

$$\varepsilon_n(A) \le \varepsilon_n^*(A), \quad n \in \mathbb{N}, \tag{22}$$

where

$$\varepsilon_n^*(A) = \frac{1}{2^{\gamma-1}} \left[I_1(A, n) + I_2(A, n)\right], \tag{23}$$

$$I_1(A, n) = \left(1 - \alpha_{n+1}\right) \int\limits_{\alpha_{n+1}}^1 \frac{(1-y)^{2\gamma-1}}{y^\gamma} \frac{\chi_n(y)dy}{1 - \alpha_{n+1} y},$$

$$I_2(A, n) = \sqrt{1 + \alpha_{n+1}^2} \int\limits_d^{\alpha_{n+1}} \frac{(1-y)^{2\gamma}}{y^\gamma \sqrt{1+y^2}} \frac{\left|\chi_n(y)\right| dy}{1 - \alpha_{n+1} y},$$

$$\chi_n(y) = \prod_{k=1}^n \frac{y - \alpha_k}{1 - \alpha_k y}, \quad \alpha_k \in [0, 1), \quad k = 1, 2, \ldots, n+1. \tag{24}$$

The inequality (21) is exact in the sense that if all the poles of approximating function have even multiplicity then this inequality becomes equality for $x = \pm 1$.

P r o o f. Assume, that parameters α_k, $k = 1, 2, \ldots, n+1$, are ordered as follows

$$0 \le d < \alpha_1 < \alpha_2 < \ldots < \alpha_{n+1} < 1.$$

From (13) and (14) it follows, that when $d\mu(t) = \varphi(t)dt$, $\varphi(t) \asymp (t-1)^\gamma$, it is natural to consider approximation $\varepsilon_n(x, A)$ as

$$\varepsilon_n(x, A) = -\frac{1}{2^{\gamma-1}} \int\limits_d^1 \sqrt{\frac{1 - 2\alpha_{n+1}\cos u + \alpha_{n+1}^2}{1 - 2y\cos u + y^2}} \frac{(1-y)^{2\gamma}}{y^\gamma} \cos\psi_n(u, y) \frac{\chi_n(y)dy}{1 - \alpha_{n+1} y}, \tag{25}$$

where $\chi_n(y)$ is defined by (24), $x = \cos u$, $\alpha_k \in [0, 1)$, $\psi_n(u, y)$ is defined by (15). Since $\left|\cos\psi_n(u, y)\right| \le 1$, we immediately get estimate (21). Now we prove its exactness. For this purpose we study the right-hand side of (25) when $x = 1$ or $u = 0$. Taking into account, that in this case $\xi = 1$, from (15) we get $\psi(0, y) = 0$. Therefore,

$$\left|\varepsilon_n(1, A)\right| = \frac{1 - \alpha_{n+1}}{2^{\gamma-1}} \int\limits_d^1 \frac{(1-y)^{2\gamma-1}}{y^\gamma} \frac{\chi_n(y)dy}{1 - \alpha_{n+1} y}. \tag{26}$$

It is not difficult to check, that the right-hand-sides of the last expression and inequality (21) for $x = 1$ coincide. Similarly, we can prove exactness of the estimate (21) for $x = -1$.

Then we are going to check the estimate (22). Keeping in mind, that $x = \cos u$, from (21) we obtain

$$\left|\varepsilon_n(x, A)\right| \le \frac{1-\alpha_{n+1}}{2^{\gamma-1}} \int_d^1 \nu(x) \frac{(1-y)^{2\gamma-1}}{y^\gamma} \frac{\left|\chi_n(y)\right| dy}{1-\alpha_{n+1} y}, \quad \nu(x) = \sqrt{\frac{1+A_1(1-x)}{1+Y(1-x)}}, \tag{27}$$

where $A_1 = \frac{2\alpha_{n+1}}{1-\alpha_{n+1}^2}$, $Y = \frac{2y}{1-y^2}$. Now we will study the function $\nu(x)$. Since

$$\nu'_x(x) = \frac{Y - A_1}{\sqrt{\left(1+A_1(1-x)\right)\left(1+Y(1-x)\right)^3}}$$

the function $\nu(x)$ decreases for $d < y < \alpha_{n+1}$ and reaches its maximum value for $x = 0$ or $u = \frac{\pi}{2}$. At the same time the function $\nu(x)$ increases for $\alpha_{n+1} < y < 1$ and reaches its maximum value for $x = 1$ or $u = 0$. Then, splitting the integral in the right-hand side of (27) into two integral over the intervals $\left[d, \alpha_{n+1}\right]$ and $\left[\alpha_{n+1}, 1\right]$, applying above said arguments, we obtain inequality (22). Theorem 3 is proved.

Some corollaries of theorem 3. Let us consider the polynomial case. Assume in (21) and (22) $\alpha_k = 0$, $k = 1, 2, \ldots, n+1$, and $\varepsilon_n(x, O) = \varepsilon_n(x)$, $\varepsilon_n(O) = \varepsilon_n$, $O = (0, 0, \ldots, 0)$ are pointwise and uniform approximations of Markov function $\hat{\mu}(x)$ by partial sums of Fourier series with respect to the Chebyshev polynomials of the first kind, provided measure $\mu(t)$ satisfies conditions of theorem 3. Then

$$\left|\varepsilon_n(x)\right| \le \frac{1}{2^{\gamma-1}} \int_d^1 \frac{(1-y)^{2\gamma}}{\sqrt{1-2y\cos u + y^2}} y^{n-\gamma} dy, \quad x = \cos u, \quad x \in [-1, 1],$$

$$\varepsilon_{2n} = \frac{1}{2^{\gamma-1}} \int_d^1 (1-y)^{2\gamma-1} y^{n-\gamma} dy, \quad n \in \mathbb{N}. \tag{28}$$

Note that, having allowed the parameter to take a zero value, the integrals in the last relations exist when the condition $n > \gamma + 1$ is satisfied.

It is interesting to study asymptotic behaviour of the integral (28) for $\gamma \in (0, +\infty) \setminus \mathbb{N}$, when $n \to \infty$. We use the Laplace's method [16; 17]. Let us write the last integral as

$$\varepsilon_n = \frac{1}{2^{\gamma-1}} \int_d^1 \frac{(1-y)^{2\gamma-1}}{y^\gamma} \mathrm{e}^{n \ln y} dy.$$

The function $\ln y$ increases when $y \in (d, 1)$ and reaches its maximum value for $y = 1$. Since $\ln y = y - 1 + o(y-1)$, $y \to 1$, and taking into account that

$$\frac{(1-y)^{2\gamma-1}}{y^\gamma} = (1-y)^{2\gamma-1} + o\left((1-y)^{2\gamma-1}\right), \quad y \to 1,$$

for small enough $\varepsilon > 0$ and $n \to \infty$ we obtain

$$\varepsilon_n \sim \frac{1}{2^{\gamma-1}} \int_{1-\varepsilon}^1 (1-y)^{2\gamma-1} \mathrm{e}^{(y-1)n} dy.$$

Applying substitution $(1-y)n = t$, finally we get

$$\varepsilon_n \sim \frac{1}{2^{\gamma-1} n^{2\gamma}} \int_0^{n\varepsilon} t^{2\gamma-1} \mathrm{e}^{-t} dt \sim \frac{\Gamma(2\gamma)}{2^{\gamma-1} n^{2\gamma}}, \quad \gamma > 0, \quad n \to \infty, \tag{29}$$

where Γ is the gamma function.

The last relationship is an asymptotic estimate of the uniform approximation of Markov function by partial sums of Fourier series with respect to the Chebyshev polynomials of the first kind provided the measure $\mu(t)$ satisfies the condition of theorem 3. It should be noted, that in this estimate we have exact constants.

Asymptotics of the majorant of uniform approximations of function $\hat{\mu}(x)$ in case of a fixed number of poles of the approximating function

Above we have found the estimates of pointwise and uniform approximations of Markov functions by rational integral operator (4) when measure $\mu(t)$, $t \in \operatorname{supp}\mu$, satisfies some conditions. It is interesting to find asymptotic expression for the quantity (23) when $n \to \infty$.

To solve this problem, we do substitution $y = \frac{1-u}{1+u}$, $dy = \frac{-2du}{(1+u)^2}$ in the integrals $I_1(A, n)$ and $I_2(A, n)$. Then

$$I_1(A, n) = 2^{2\gamma}\beta_{n+1}\int_0^{\beta_{n+1}} \frac{u^{2\gamma-1}}{\left(1-u^2\right)^{\gamma}} \prod_{k=1}^{n} \frac{\beta_k - u}{\beta_k + u} \frac{du}{\beta_{n+1}+u},$$

$$I_2(A, n) = 2^{2\gamma-1}\int_{\beta_{n+1}}^{D} \frac{\sqrt{1+\beta_{n+1}^2}\,u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}} \left|\prod_{k=1}^{n} \frac{u-\beta_k}{u+\beta_k}\right| \frac{du}{\beta_{n+1}+u},$$

where

$$D = \frac{1-d}{1+d},\ D \in (0, 1],\ \beta_k = \frac{1-\alpha_k}{1+\alpha_k},\ \beta_k \in (0, 1],\ k = 1, 2, \ldots, n+1. \tag{30}$$

Note that, if $D = 1$ and $\beta_k \neq 1$, $k = 1, 2, \ldots, n$, then $\gamma \in (0, 1)$. Otherwise, the integral $I_2(A, n)$ obviously diverges.

Let q be a natural number, $0 < q < n$, and $A_{n,q}$ be a set of parameters $\alpha \in A$ such, that there are exactly q distinct numbers among $\alpha_1, \alpha_2, \ldots, \alpha_n$ and multiplicity of each parameter is equal to $m = \frac{n}{q}$, $n = mq$.

It should be noted, that we choose appropriate set $A_{n,q}$ of parameters α_k, $k = 1, 2, \ldots, q$, for each particular value of n. In other words $\alpha_k = \alpha_k(m)$, $m = 1, 2, \ldots$. In this case we assume, that parameters α_k, $k = 1, 2, \ldots, q$, satisfy the condition

$$m\sum_{k=1}^{q}(1-\alpha_k) \xrightarrow[m\to\infty]{} \infty.$$

Theorem 4. *For any natural numbers n and q (q is fixed and* $0 < q < n$*) the following asymptotic equality holds*

$$\varepsilon_n^*\left(A_{n,q}\right) \sim \frac{2^{1-\gamma}\Gamma(2\gamma)}{\left(m\sum_{k=1}^{q}\frac{1}{\beta_k}\right)^{2\gamma}} + \frac{\beta_{n+1}^{2\gamma-1}}{2^{2-\gamma}m\left(1-\beta_{n+1}^2\right)^{\gamma}\sum_{k=1}^{q}\frac{\beta_k}{\beta_k^2-\beta_{n+1}^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-\beta_{n+1}}{\beta_k+\beta_{n+1}}\right)^m +$$

$$+\,2^{\gamma-\frac{1}{2}}\sqrt{\frac{\pi\left(1+\beta_{n+1}^2\right)}{m}}\sum_{j=1}^{q-1}\frac{b_j^{2\gamma-\frac{1}{2}}}{\left(1-b_j^2\right)^{\gamma}\sqrt{1+b_j^2}\left(\beta_{n+1}+b_j\right)}\left(\prod_{k=1}^{j}\frac{\beta_k-b_j}{\beta_k+b_j}\prod_{k=j+1}^{q}\frac{b_j-\beta_k}{b_j+\beta_k}\right)^m \times$$

$$\times\frac{1}{\sqrt{\sum_{k=1}^{j}\frac{\beta_k}{\left(\beta_k^2-b_j^2\right)^2}+\sum_{k=j+1}^{q}\frac{\beta_k}{\left(b_j^2-\beta_k^2\right)^2}}} + \Phi\left(A_{n,q}, n\right),\ m\to\infty, \tag{31}$$

where

$$\Phi\left(A_{n,q}, n\right) = \begin{cases} \dfrac{\sqrt{1+\beta_{n+1}^2}\,\Gamma(1-\gamma)}{\sqrt{2}\left(1+\beta_{n+1}\right)\left(2m\sum\limits_{k=1}^{q}\dfrac{\beta_k}{1-\beta_k^2}\right)^{1-\gamma}}\left(\prod\limits_{k=1}^{q}\dfrac{1-\beta_k}{1+\beta_k}\right)^m, \ D = 1, \\ \dfrac{\sqrt{1+\beta_{n+1}^2}\,D^{2\gamma}\left(1-D^2\right)^{1-2\gamma}}{2^{1-\gamma}m\sqrt{1-D^4}\left(\beta_{n+1}+D\right)\sum\limits_{k=1}^{q}\dfrac{\beta_k}{D^2-\beta_k^2}}\left(\prod\limits_{k=1}^{q}\dfrac{D-\beta_k}{D+\beta_k}\right)^m, \ D \neq 1, \end{cases} \tag{32}$$

$b_j \in \left(\beta_{j+1}, \beta_j\right)$, $j = 1, 2, \ldots, q-1$, *is the only root of the equation*

$$-\sum_{k=1}^{j}\frac{\beta_k}{\beta_k^2-u^2} + \sum_{k=j+1}^{q}\frac{\beta_k}{u^2-\beta_k^2} = 0, \tag{33}$$

Γ *is the gamma function.*

P r o o f. In order to obtain formula (32) we need to consider each of the integrals $I_1\left(A_{n,q}, n\right)$ and $I_2\left(A_{n,q}, n\right)$ from (23) separately. We are going to study their asymptotic behaviour when $m \to \infty$, $n = mq$. For this purpose, we need two lemmas.

Lemma 4. *The following asymptotic equality holds*

$$I_1\left(A_{n,q}, n\right) \sim \frac{\Gamma(2\gamma)}{\left(m\sum\limits_{k=1}^{q}\dfrac{1}{\beta_k}\right)^{2\gamma}}, \ m \to \infty, \tag{34}$$

where Γ *is the gamma function.*

P r o o f. It is clear, that under the conditions of theorem 4, the integral can be written as follows

$$I_1\left(A_{n,q}, n\right) = 2^{2\gamma}\beta_{n+1}\int_0^{\beta_{n+1}}\frac{u^{2\gamma-1}}{\left(1-u^2\right)^{\gamma}}\left(\prod_{k=1}^{q}\frac{\beta_k-u}{\beta_k+u}\right)^m\frac{du}{\beta_{n+1}+u}.$$

To study its asymptotic behaviour, we use Laplace's method [16; 17]. Transform this integral to the form, at which the indicated method can be implemented. We have

$$I_1\left(A_{n,q}, n\right) = 2^{2\gamma}\beta_{n+1}\int_0^{\beta_{n+1}}\frac{u^{2\gamma-1}}{\left(1-u^2\right)^{\gamma}}e^{mS(u)}\frac{du}{\beta_{n+1}+u}, \ S(u) = \sum_{k=1}^{q}\ln\frac{\beta_k-u}{\beta_k+u}.$$

Since $S'(u) = -2\sum\limits_{k=1}^{q}\dfrac{\beta_k}{\beta_k^2-u^2} < 0$, the function $S(u)$ decreases on the interval $\left[0, \beta_{n+1}\right]$ and reaches its maximum value at $u = 0$. Hence, for $m \to \infty$ the value of the initial integral is approximately equal to the value of this integral over the small interval $[0, \varepsilon]$. On this interval we can replace functions by linear ones

$$S(u) \approx -2u\sum_{k=1}^{q}\frac{1}{\beta_k}, \ f(u) = \frac{u^{2\gamma-1}}{\left(1-u^2\right)^{\gamma}}\frac{1}{\beta_{n+1}+u} \approx \frac{1}{\beta_{n+1}}u^{2\gamma-1}.$$

Then

$$I_1\left(A_{n,q}, n\right) \sim 2^{2\gamma}\int_0^{\varepsilon}u^{2\gamma-1}\exp\left[-2mu\sum_{k=1}^{q}\frac{1}{\beta_k}\right]du, \ m \to \infty.$$

In the last integral we do substitution $2mu\sum\limits_{k=1}^{q}\dfrac{1}{\beta_k} = t$. Therefore,

$$I_1\left(A_{n,q},\, n\right) \sim 2^{2\gamma} \frac{1}{2^{2\gamma}\left(\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^{2\gamma} m^{2\gamma}} \int\limits_0^{\varphi(m,\,\varepsilon)} t^{2\gamma-1}\mathrm{e}^{-t}dt,\ m\to\infty,$$

where $\varphi(m,\,\varepsilon) = \varepsilon m\sum\limits_{k=1}^{q}\frac{1}{\beta_k} \to \infty$, when $m\to\infty$. The last asymptotic equality immediately leads to (34). Lemma 4 is proved.

Lemma 5. *The following asymptotic equality holds*

$$I_2\left(A_{n,q},\, n\right) \sim \frac{2^{2\gamma-3}\beta_{n+1}^{2\gamma-1}}{m\left(1-\beta_{n+1}^2\right)^{\gamma}\sum\limits_{k=1}^{q}\frac{\beta_k}{\beta_k^2-\beta_{n+1}^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-\beta_{n+1}}{\beta_k+\beta_{n+1}}\right)^m +$$

$$+\, 2^{2\gamma-\frac{3}{2}}\sqrt{\frac{\pi\left(1+\beta_{n+1}^2\right)}{m}}\sum_{j=1}^{q-1}\frac{b_j^{2\gamma-\frac{1}{2}}}{\left(1-b_j^2\right)^{\gamma}\sqrt{1+b_j^2}\left(\beta_{n+1}+b_j\right)}\left(\prod_{k=1}^{j}\frac{\beta_k-b_j}{\beta_k+b_j}\prod_{k=j+1}^{q}\frac{b_j-\beta_k}{b_j+\beta_k}\right)^m \times$$

$$\times \frac{1}{\sqrt{\sum\limits_{k=1}^{j}\frac{\beta_k}{\left(\beta_k^2-b_j^2\right)^2}+\sum\limits_{k=j+1}^{q}\frac{\beta_k}{\left(b_j^2-\beta_k^2\right)^2}}} + \frac{1}{2^{1-\gamma}}\Phi\left(A_{n,q},\, n\right),\ m\to\infty, \tag{35}$$

where $\Phi\left(A_{n,q},\, n\right)$ *is defined in (32).*

P r o o f. Under the conditions of theorem 4, the integral can be written as follows

$$I_2\left(A_{n,q},\, n\right) = 2^{2\gamma-1}\sqrt{1+\beta_{n+1}^2}\int\limits_{\beta_{n+1}}^{D}\frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}}\left|\prod_{k=1}^{q}\frac{u-\beta_k}{u+\beta_k}\right|^m\frac{du}{\beta_{n+1}+u},$$

where D is defined in (30). Denote

$$I_3\left(A_{n,q},\, n\right) = \int\limits_{\beta_{n+1}}^{\beta_q}\frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-u}{\beta_k+u}\right)^m\frac{du}{\beta_{n+1}+u},$$

$$I_4\left(A_{n,q},\, n\right) = \sum_{j=1}^{q-1}\int\limits_{\beta_{j+1}}^{\beta_j}\frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}}\left(\prod_{k=1}^{j}\frac{\beta_k-u}{\beta_k+u}\prod_{k=j+1}^{q}\frac{u-\beta_k}{u+\beta_k}\right)^m\frac{du}{\beta_{n+1}+u},$$

$$I_5\left(A_{n,q},\, n\right) = \int\limits_{\beta_1}^{D}\frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}}\left(\prod_{k=1}^{q}\frac{u-\beta_k}{u+\beta_k}\right)^m\frac{du}{\beta_{n+1}+u}.$$

Thus,

$$I_2\left(A_{n,q},\, n\right) = 2^{2\gamma-1}\sqrt{1+\beta_{n+1}^2}\left[I_3\left(A_{n,q},\, n\right)+I_4\left(A_{n,q},\, n\right)+I_5\left(A_{n,q},\, n\right)\right]. \tag{36}$$

To study asymptotic behaviour of these integrals, we use Laplace's method again. So, for the integral $I_3\left(A_{n,q},\, n\right)$ we have

$$I_3\left(A_{n,q},\, n\right) = \int\limits_{\beta_{n+1}}^{\beta_q} f(u)\mathrm{e}^{mS(u)}du,\ f(u) = \frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}\left(\beta_{n+1}+u\right)},\ S(u) = \sum_{k=1}^{q}\ln\frac{\beta_k-u}{\beta_k+u}.$$

The function the function $S(u)$ decreases on the interval $\left[\beta_{n+1}, \beta_q\right]$, and reaches its maximum value at $u = \beta_{n+1}$. Expanding this function into Taylor series at the point $u = \beta_{n+1}$, and using decomposition

$$f(u) = \frac{\beta_{n+1}^{2\gamma-1}}{2\left(1-\beta_{n+1}^2\right)^{\gamma}\sqrt{1+\beta_{n+1}^2}}, \ u \to \beta_{n+1},$$

for small enough $\varepsilon > 0$ and $m \to \infty$ we find

$$I_3\left(A_{n,q}, n\right) \sim \frac{\beta_{n+1}^{2\gamma-1}}{2\left(1-\beta_{n+1}^2\right)^{\gamma}\sqrt{1+\beta_{n+1}^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-\beta_{n+1}}{\beta_k+\beta_{n+1}}\right)^{m}\int\limits_{\beta_{n+1}}^{\beta_{n+1}+\varepsilon}\exp\left[-2m\sum_{k=1}^{q}\frac{\beta_k}{\beta_k^2-\beta_{n+1}^2}\left(u-\beta_{n+1}\right)\right]du.$$

From here we get

$$I_3\left(A_{n,q}, n\right) \sim \frac{\beta_{n+1}^{2\gamma-1}}{4m\left(1-\beta_{n+1}^2\right)^{\gamma}\sqrt{1+\beta_{n+1}^2}\sum\limits_{k=1}^{q}\frac{\beta_k}{\beta_k^2-\beta_{n+1}^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-\beta_{n+1}}{\beta_k+\beta_{n+1}}\right)^{m}, \ m \to \infty. \tag{37}$$

Consider the integral $I_4\left(A_{n,q}, n\right)$. We have

$$I_4\left(A_{n,q}, n\right) = \sum_{j=1}^{q-1}\int\limits_{\beta_{j+1}}^{\beta_j} f(u)\mathrm{e}^{mS(u)}du,$$

$$f(u) = \frac{u^{2\gamma}}{\left(1-u^2\right)^{\gamma}\sqrt{1+u^2}\left(\beta_{n+1}+u\right)}, \ S(u) = \sum_{k=1}^{j}\ln\frac{\beta_k-u}{\beta_k+u} + \sum_{k=j+1}^{q}\ln\frac{u-\beta_k}{u+\beta_k}.$$

Since

$$S'(u) = \sum_{k=1}^{j}\frac{-2\beta_k}{\beta_k^2-u^2} + \sum_{k=j+1}^{q}\frac{2\beta_k}{u^2-\beta_k^2}, \ S''(u) = \sum_{k=1}^{j}\frac{-4\beta_k u}{\left(\beta_k^2-u^2\right)^2} + \sum_{k=j+1}^{q}\frac{-4\beta_k u}{\left(u^2-\beta_k^2\right)^2} < 0,$$

there exist an inner point $b_j \in \left(\beta_{j+1}, \beta_j\right)$ such that the function $S(u)$ reaches its maximum value at this point. Besides $S'\left(b_j\right) = 0$. Using decompositions

$$S(u) = \sum_{k=1}^{j}\ln\frac{\beta_k-b_j}{\beta_k+b_j} + \sum_{k=j+1}^{q}\ln\frac{b_j-\beta_k}{b_j+\beta_k} - \left[\sum_{k=1}^{j}\frac{2b_j\beta_k}{\left(\beta_k^2-b_j^2\right)^2} + \sum_{k=j+1}^{q}\frac{2b_j\beta_k}{\left(b_j^2-\beta_k^2\right)^2}\right]\left(u-b_j\right)^2 + o\left(\left(u-b_j\right)^2\right),$$

$$f(u) = \frac{b_j^{2\gamma}}{\left(1-b_j^2\right)^{\gamma}\sqrt{1+b_j^2}\left(\beta_{n+1}+b_j\right)},$$

when $u \to b_j$, for small enough $\varepsilon > 0$ and $m \to \infty$ we find

$$I_4\left(A_{n,q}, n\right) \sim \sum_{j=1}^{q-1}\frac{b_j^{2\gamma}}{\left(1-b_j^2\right)^{\gamma}\sqrt{1+b_j^2}\left(\beta_{n+1}+b_j\right)}\left(\prod_{k=1}^{j}\frac{\beta_k-b_j}{\beta_k+b_j}\prod_{k=j+1}^{q}\frac{b_j-\beta_k}{b_j+\beta_k}\right)^{m} \times$$

$$\times\int\limits_{-\varepsilon}^{\varepsilon}\exp\left[-2b_j m\left(\sum_{k=1}^{j}\frac{\beta_k}{\left(\beta_k^2-b_j^2\right)^2} + \sum_{k=j+1}^{q}\frac{\beta_k}{\left(b_j^2-\beta_k^2\right)^2}\right)u^2\right]du.$$

Taking into account, that

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi},$$

from the last asymptotic equality for $m \to \infty$ we obtain

$$I_4\left(A_{n,q}, n\right) \sim \sqrt{\frac{\pi}{2m}} \sum_{j=1}^{q-1} \frac{b_j^{2\gamma - \frac{1}{2}} \left(\prod_{k=1}^{j} \frac{\beta_k - b_j}{\beta_k + b_j} \prod_{k=j+1}^{q} \frac{b_j - \beta_k}{b_j + \beta_k} \right)^m}{\left(1 - b_j^2\right)^{\gamma} \sqrt{1 + b_j^2} \left(\beta_{n+1} + b_j\right) \sqrt{\sum_{k=1}^{j} \frac{\beta_k}{\left(\beta_k^2 - b_j^2\right)^2} + \sum_{k=j+1}^{q} \frac{\beta_k}{\left(b_j^2 - \beta_k^2\right)^2}}}. \tag{38}$$

Consider the integral $I_5\left(A_{n,q}, n\right)$. Let $D \neq 1$ (see (30)). Change variables $u = \cos\theta$. Then

$$I_5\left(A_{n,q}, n\right) = \int_{\arccos D}^{\arccos \beta_1} f(\theta) e^{mS(\theta)} d\theta, \ f(\theta) = \frac{\cos^{2\gamma}\theta \sin^{1-2\gamma}\theta}{\sqrt{1 + \cos^2\theta}\left(\beta_{n+1} + \cos\theta\right)}, \ S(\theta) = \sum_{k=1}^{q} \ln \frac{\cos\theta - \beta_k}{\cos\theta + \beta_k}.$$

Since

$$S'(\theta) = \sum_{k=1}^{q} \frac{-2\beta_k \sin\theta}{\cos^2\theta - \beta_k^2} < 0,$$

the function decreases for $\theta \in \left(\arccos D, \arccos \beta_1\right)$ and reaches its maximum value at the point $\theta = \arccos D$. Applying decompositions when $\theta \to \arccos D$

$$S(\theta) = \sum_{k=1}^{q} \ln \frac{D - \beta_k}{D + \beta_k} - 2\sqrt{1 - D^2} \sum_{k=1}^{q} \frac{\beta_k}{D^2 - \beta_k^2} (\theta - \arccos D), \ f(\theta) = \frac{D^{2\gamma}\left(1 - D^2\right)^{1-2\gamma}}{\sqrt{1 + D^2}\left(\beta_{n+1} + D\right)},$$

for small enough $\varepsilon > 0$ and $m \to \infty$ we find

$$I_5\left(A_{n,q}, n\right) \sim \frac{D^{2\gamma}\left(1 - D^2\right)^{1-2\gamma}}{\sqrt{1 + D^2}\left(\beta_{n+1} + D\right)} \left(\prod_{k=1}^{q} \frac{D - \beta_k}{D + \beta_k} \right)^m \int_0^{\varepsilon} \exp\left[-2\sqrt{1 - D^2}\, m \sum_{k=1}^{q} \frac{\beta_k}{D^2 - \beta_k^2} \theta \right] d\theta.$$

This formula leads to the main term of the asymptotics

$$I_5\left(A_{n,q}, n\right) \sim \frac{D^{2\gamma}\left(1 - D^2\right)^{1-2\gamma}}{2m\sqrt{1 - D^4}\left(\beta_{n+1} + D\right) \sum_{k=1}^{q} \frac{\beta_k}{D^2 - \beta_k^2}} \left(\prod_{k=1}^{q} \frac{D - \beta_k}{D + \beta_k} \right)^m, \ m \to \infty. \tag{39}$$

If $D = 1$, then in a similar way we obtain asymptotic equality

$$I_5\left(A_{n,q}, n\right) \sim \frac{1}{\sqrt{2}\left(1 + \beta_{n+1}\right) \left(m \sum_{k=1}^{q} \frac{\beta_k}{1 - \beta_k^2} \right)^{1-\gamma}} \left(\prod_{k=1}^{q} \frac{1 - \beta_k}{1 + \beta_k} \right)^m \int_0^{\varphi(\varepsilon, m)} \theta^{1-2\gamma} e^{-\theta^2} d\theta, \ m \to \infty,$$

where $\varphi(\varepsilon, m) = \varepsilon \sqrt{m \sum_{k=1}^{q} \frac{\beta_k}{1 - \beta_k^2}} \to \infty$ for $m \to \infty$. Taking into account that

$$\int_0^{+\infty} \theta^{1-2\gamma} e^{-\theta^2} d\theta = \frac{1}{2} \Gamma(1 - \gamma), \ \gamma \in [0, 1),$$

we finally have

$$I_5\left(A_{n,q}, n\right) \sim \frac{\Gamma(1-\gamma)}{2\sqrt{2}\left(1+\beta_{n+1}\right)\left(m\sum\limits_{k=1}^{q}\frac{\beta_k}{1-\beta_k^2}\right)^{1-\gamma}}\left(\prod_{k=1}^{q}\frac{1-\beta_k}{1+\beta_k}\right)^m, \quad m\to\infty. \tag{40}$$

Plugging asymptotic equalities (37)–(40) into (36), we get (35). Lemma 5 is proved.

Now we return to the proof of theorem 4. Using (34), (35) and (23) we immediately obtain (31). This concludes the proof of theorem 4.

Order estimate of the uniform approximation of function $\hat{\mu}(x)$ in case of fixed number of poles of approximating function

In this section we consider the problem of minimizing the right-hand side of (31) choosing an optimal set of parameters $A_{n,q}^* = \left\{\alpha^* : \alpha^* = \left(\alpha_1^*, \alpha_2^*, \ldots, \alpha_q^*\right)\right\}$. In other words, we will find an estimate of the best uniform approximation of the Markov function by the rational integral operator (4), when measure $\mu(t)$ satisfies conditions of theorem 3. Let

$$\varepsilon_{n,q} = \inf_{\alpha\in[0,1)}\varepsilon_n\left(A_{n,q}\right), \quad \varepsilon_{n,q}^* = \inf_{\alpha\in[0,1)}\varepsilon_n^*\left(A_{n,q}\right),$$

where $\varepsilon_n\left(A_{n,q}\right)$ is uniform approximation of Markov function by the rational integral operator (4) in case of q geometrically distinct poles of approximation function. Obviously, from (22) it follows, that

$$\varepsilon_{n,q} \le \varepsilon_{n,q}^*, \quad n\in\mathbb{N}.$$

Theorem 5. *Let measure $\mu(t)$ satisfy conditions of theorem 3. Then for any natural numbers n and q (q is fixed and $0 < q < n$) the following asymptotic equalities hold:*

1) $\varepsilon_{n,q}^* \sim 2^{1+\gamma}\left(q^{2q+1}D\gamma^{2q-1}\right)^{2\gamma}\Gamma(2\gamma)\left(\prod\limits_{k=1}^{q-1}(q-k)^2\right)^{2\gamma}\left(\dfrac{\ln^{2q-1} n}{n^{2q}}\right)^{2\gamma}, \ D\in(0,1], \ n\to\infty;$

2) if n is even, then $\varepsilon_{n,q} \sim \varepsilon_{n,q}^*, \ n\to\infty,$

where Γ is the gamma function.

P r o o f. Let the set of parameters $A_{n,q} = \left\{\alpha : \alpha = \left(\alpha_1, \alpha_2, \ldots, \alpha_q\right)\right\}$ be defined as follows

$$\alpha_k = \frac{1-\beta_k}{1+\beta_k}, \quad \beta_k = c_k\left(\frac{\ln m}{m}\right)^{2k-1}, \quad k = 1, 2, \ldots, q.$$

Also assume, that $\beta_{n+1} = \beta_q$. We are going to study the right-hand side of the asymptotic equality (31) in this case. So, for the first term we find, that

$$S_1\left(A_{n,q}\right) = \frac{2^{1-\gamma}\Gamma(2\gamma)}{\left(m\sum\limits_{k=1}^{q}\frac{1}{c_k}\left(\frac{m}{\ln m}\right)^{2k-1}\right)^{2\gamma}} \sim \frac{2^{1-\gamma}\Gamma(2\gamma)c_q^{2\gamma}\ln^{4q\gamma-2\gamma} m}{m^{4q\gamma}}, \quad m\to\infty. \tag{41}$$

Since $\beta_{n+1} = \beta_q$, it is no difficult to show, that the second term in the right-hand side of (31) equals to zero.

Let us study asymptotics of the third term when $m\to\infty$. We have

$$S_3\left(A_{n,q}\right) = 2^{\gamma-\frac{1}{2}}\sqrt{\frac{\pi\left(1+\beta_{n+1}^2\right)}{m}}\sum_{j=1}^{q-1}\frac{b_j^{2\gamma-\frac{3}{2}}}{\left(1-b_j^2\right)^{\gamma}\left(1+\dfrac{\beta_{n+1}}{b_j}\right)}\left(\prod_{k=1}^{j}\frac{\beta_k-b_j}{\beta_k+b_j}\prod_{k=j+1}^{q}\frac{b_j-\beta_k}{b_j+\beta_k}\right)^m \times$$

$$\times\frac{1}{\sqrt{\left(1+b_j^2\right)\left(\sum\limits_{k=1}^{j}\dfrac{\beta_k}{\left(\beta_k^2-b_j^2\right)^2}+\sum\limits_{k=j+1}^{q}\dfrac{\beta_k}{\left(b_j^2-\beta_k^2\right)^2}\right)}}. \tag{42}$$

For the further proof we need to figure out properties of the parameters b_j, $b_j \in \left(\beta_{j+1}, \beta_j\right)$, $j = 1, 2, \ldots, q-1$, which are the roots of the equation (33). For this purpose, we prove the auxiliary statement.

Lemma 6. *If parameters* β_k, $k = 1, 2, \ldots, q$, *satisfy the condition*

$$\beta_k = c_k \left(\frac{\ln m}{m}\right)^{2k-1}, \; k = 1, 2, \ldots, q,$$

then for the roots of the equation (33) the following asymptotic equality holds

$$b_j \sim \sqrt{c_j c_{j+1}} \left(\frac{\ln m}{m}\right)^{2j}, \; j = 1, 2, \ldots, q-1, \; m \to \infty.$$

P r o o f. It is clear, that in this case the equation (33) can be written as follows

$$\sum_{k=1}^{j} \frac{1}{\beta_k \left(1 - \dfrac{u}{\beta_k}\right)^2} = \frac{1}{u^2} \sum_{k=j+1}^{q} \frac{\beta_k}{1 - \left(\dfrac{\beta_k}{u}\right)^2}, \; j = 1, 2, \ldots, q-1. \tag{43}$$

For each fixed $j = 1, 2, \ldots, q-1$ the root b_j satisfies the double inequality

$$c_{j+1} \left(\frac{\ln m}{m}\right)^{2j+1} < b_j < c_j \left(\frac{\ln m}{m}\right)^{2j-1}, \; j = 1, 2, \ldots, q-1.$$

Hence, for given values of the parameters β_k, $k = 1, 2, \ldots, q$, and $m \to \infty$ we find that

$$\sum_{k=1}^{j} \frac{1}{\beta_k \left(1 - \dfrac{b_j}{\beta_k}\right)^2} \sim \sum_{k=1}^{j} \frac{1}{\beta_k} \sim \frac{1}{c_j} \left(\frac{m}{\ln m}\right)^{2j-1}.$$

In a similar way,

$$\frac{1}{b_j^2} \sum_{k=j+1}^{q} \frac{\beta_k}{1 - \left(\dfrac{b_j}{\beta_k}\right)^2} \sim \frac{1}{b_j^2} \sum_{k=j+1}^{q} \beta_k \sim \frac{c_{j+1}}{b_j^2} \left(\frac{\ln m}{m}\right)^{2j+1}.$$

Plugging last two asymptotic equalities into (43), we obtain

$$\frac{1}{c_j} \left(\frac{m}{\ln m}\right)^{2j-1} \sim \frac{c_{j+1}}{b_j^2} \left(\frac{\ln m}{m}\right)^{2j+1}, \; m \to \infty.$$

Statement of lemma 6 immediately follows from the last formula.

Now we continue to study asymptotic behaviour of the sum $S_3\left(A_{n,q}\right)$ (see (42)). Taking into account the result of lemma 6, we have

$$\left(\prod_{k=1}^{j} \frac{\beta_k - b_j}{\beta_k + b_j} \prod_{k=j+1}^{q} \frac{b_j - \beta_k}{b_j + \beta_k}\right)^m \sim \exp\left[-2m\left(\sum_{k=1}^{j} \frac{b_j}{\beta_k} + \sum_{k=j+1}^{q} \frac{\beta_k}{b_j}\right)\right] \sim \frac{1}{m^{4\sqrt{\frac{c_{j+1}}{c_j}}}}, \; m \to \infty.$$

Also

$$\frac{1}{\sqrt{\displaystyle\sum_{k=1}^{j} \frac{\beta_k}{\left(\beta_k^2 - b_j^2\right)^2} + \sum_{k=j+1}^{q} \frac{\beta_k}{\left(b_j^2 - \beta_k^2\right)^2}}} \sim \frac{1}{\sqrt{\displaystyle\sum_{k=1}^{j} \frac{1}{\beta_k^3} + \sum_{k=j+1}^{q} \frac{\beta_k}{b_j^4}}} \sim \frac{c_j \sqrt{c_{j+1}} \ln^{3j-\frac{1}{2}} m}{m^{3j-\frac{1}{2}}}, \; m \to \infty.$$

Therefore,

$$S_3\left(A_{n,q}\right) \sim 2^{\gamma-\frac{1}{2}}\sqrt{\frac{\pi}{m}}\sum_{j=1}^{q-1}\left(\sqrt{c_j c_{j+1}}\right)^{2\gamma-\frac{3}{2}}\left(\frac{\ln m}{m}\right)^{4j\gamma-3j}\frac{c_j\sqrt{c_{j+1}}\ln^{3j-\frac{1}{2}} m}{m^{4\sqrt{\frac{c_{j+1}}{c_j}}+3j-\frac{1}{2}}} =$$

$$= 2^{\gamma-\frac{1}{2}}\sqrt{\pi}\sum_{j=1}^{q-1}\frac{c_j^{\gamma+\frac{1}{4}}c_{j+1}^{\gamma-\frac{1}{4}}\ln^{4j\gamma-\frac{1}{2}} m}{m^{4j\gamma+4\sqrt{\frac{c_{j+1}}{c_j}}}},\ m\to\infty. \tag{44}$$

Similarly, using (32) we see, that

$$\Phi\left(A_{n,q},\ n\right) \sim \begin{cases} \dfrac{\Gamma(1-\gamma)}{\sqrt{2}\left(2c_1\ln m\right)^{1-\gamma} m^{2c_1}},\ D=1, \\ \dfrac{D^{2\gamma-1}\left(1-D^2\right)^{1-2\gamma}}{2^{1-\gamma}\sqrt{1-D^4}\,c_1 m^{2\frac{c_1}{D}}\ln m},\ D\neq 1,\ m\to\infty. \end{cases} \tag{45}$$

Plugging (41), (44) and (45) into (31), we obtain for $m\to\infty$

$$\varepsilon_n^*\left(A_{n,q}\right) \sim \frac{2^{1-\gamma}\Gamma(2\gamma)c_q^{2\gamma}\ln^{4q\gamma-2\gamma} m}{m^{4q\gamma}} + 2^{\gamma-\frac{1}{2}}\sqrt{\pi}\sum_{j=1}^{q-1}\frac{c_j^{\gamma+\frac{1}{4}}c_{j+1}^{\gamma-\frac{1}{4}}\ln^{4j\gamma-\frac{1}{2}} m}{m^{4j\gamma+4\sqrt{\frac{c_{j+1}}{c_j}}}} + \Phi\left(A_{n,q},\ n\right).$$

If the parameters c_k, $k = 1, 2, \ldots, q$, satisfy the condition

$$\begin{cases} 4q\gamma = 4j\gamma + \sqrt{\dfrac{c_{j+1}}{c_j}},\ j = 1, 2, \ldots, q-1, \\ 4q\gamma = 2c_1,\ D = 1, \\ 4q\gamma = \dfrac{2c_1}{D},\ D\neq 1, \end{cases} \tag{46}$$

then it is not difficult to get

$$c_q = 2qD\gamma^{2q-1}\prod_{k=1}^{q-1}(q-k)^2.$$

Besides, in the last asymptotic equality we have

$$\varepsilon_n^*\left(A_{n,q}\right) = 2^{1+\gamma}\left(q^{2q+1}D\gamma^{2q-1}\right)^{2\gamma}\Gamma(2\gamma)\left(\prod_{k=1}^{q-1}(q-k)^2\right)^{2\gamma}\times$$
$$\times\left(\frac{\ln^{2q-1} n}{n^{2q}}\right)^{2\gamma} + o\left(\frac{\ln^{4q\gamma-2\gamma} n}{n^{4q\gamma}}\right),\ n\to\infty. \tag{47}$$

Using the considerations proposed in [23; 24], it is easy to show that it is precisely with the found c_k, $k = 1, 2, \ldots, q$, the set of parameters β_k, $k = 1, 2, \ldots, q$, is optimal in the sense that the quantity $\varepsilon_n^*\left(A_{n,q}\right)$ has an asymptotically minimal value. This proves the first relation in theorem 6.

To check the second statement of theorem 6 we use the fact, that the estimate (21) is exact at the points $x = \pm 1$ for even n, $n = mq$. In this case from (26) we find

$$\left|\varepsilon_n\left(1,\ A_{n,q}\right)\right| = \frac{1-\alpha_q}{2^{\gamma-1}}\int_d^1 \frac{(1-y)^{2\gamma-1}}{y^\gamma}\left(\prod_{k=1}^{q}\frac{y-\alpha_k}{1-\alpha_k y}\right)^m \frac{dy}{1-\alpha_q y}.$$

Changing variables $y = \frac{1-u}{1+u},\ dy = \frac{-2du}{(1+u)^2}$, in the right-hand side of this relationship, we obtain

$$\left|\varepsilon_n\left(\pm 1,\ A_{n,q}\right)\right| = 2^{\gamma+1}\beta_q \int_0^D \frac{u^{2\gamma-1}}{\left(1-u^2\right)^\gamma}\left(\prod_{k=1}^{q}\frac{\beta_k-u}{\beta_k+u}\right)^m \frac{du}{\beta_q+u},\ D = \frac{1-d^2}{1+d^2}.$$

Applying the well-known techniques for studying asymptotic behaviour to the last integral, we obtain

$$\left|\varepsilon_n\left(\pm 1,\ A_{n,q}\right)\right| \sim \frac{2^{1-\gamma}\Gamma(2\gamma)}{\left(m\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^{2\gamma}} + \frac{\beta_q\beta_{n+1}^{2\gamma-1}}{2^{2-\gamma}m\left(1-\beta_{n+1}^2\right)^\gamma \sum\limits_{k=1}^{q}\frac{\beta_k}{\beta_k^2-\beta_{n+1}^2}}\left(\prod_{k=1}^{q}\frac{\beta_k-\beta_{n+1}}{\beta_k+\beta_{n+1}}\right)^m +$$

$$+\,2^{\gamma-\frac{1}{2}}\beta_q\sqrt{\frac{\pi\left(1+\beta_{n+1}^2\right)}{m}}\sum_{j=1}^{q-1}\frac{b_j^{2\gamma-\frac{1}{2}}}{\left(1-b_j^2\right)^\gamma\sqrt{1+b_j^2}\left(\beta_{n+1}+b_j\right)}\left(\prod_{k=1}^{j}\frac{\beta_k-b_j}{\beta_k+b_j}\prod_{k=j+1}^{q}\frac{b_j-\beta_k}{b_j+\beta_k}\right)^m \times$$

$$\times\frac{1}{\sqrt{\sum\limits_{k=1}^{j}\frac{\beta_k}{\left(\beta_k^2-b_j^2\right)^2}+\sum\limits_{k=j+1}^{q}\frac{\beta_k}{\left(b_j^2-\beta_k^2\right)^2}}} + \beta_q\Phi\left(A_{n,q},\ n\right),\ m\to\infty,$$

where $\Phi\left(A_{n,q},\ n\right)$ is defined in (32). We note that the right-hand side of the last asymptotic equality differs from (31) in that the second, third, and fourth terms are multiplied by β_q. Therefore, if the parameters βk, $k = 1, 2, \ldots, q$, in its right-hand side are optimal with coefficients calculated by formulas (46), then it is easy to obtain

$$\left|\varepsilon_n\left(\pm 1,\ A_{n,q}^*\right)\right| \sim \varepsilon_{n,q}^*,\ n\to\infty,$$

where is defined by the first statement of theorem 5. Hence, we conclude that for even n the estimate of uniform approximations (47) is asymptotically attainable at the points $x = \pm 1$. Therefore, under the conditions of theorem 3, it is the norm of approximation of Markov functions by rational integral operator (4). Theorem 5 is proved.

Remark 4. From theorem 5 if follows, that for uniform approximation of Markov function provided $d\mu(t) = \varphi(t)dt$ and $\varphi(t) \asymp (t-1)^\gamma$ by rational integral operator (4) in case of q geometrically distinct poles of approximating function, the following asymptotic expression holds

$$\limsup_{n\to\infty}\left(\frac{n^{2q}}{\ln^{2q-1}n}\right)^{2\gamma}\varepsilon_{n,q} = 2^{1+\gamma}\left(q^{2q+1}D\gamma^{2q-1}\right)^{2\gamma}\Gamma(2\gamma)\left(\prod_{k=1}^{q-1}(q-k)^2\right)^{2\gamma}.$$

Similar by order estimate was obtained in [15] for of approximation by partial sums of Fourier series with respect to the system of rational functions, introduced by M. M. Dzhrbashyan and A. A. Kitbalyan [12] with q geometrically distinct poles of approximating function.

Corollary 3 (case of one fixed pole). *Under conditions of theorem 3 in case of one fixed pole of approximating function the following relations hold:*

1) $\varepsilon_{n,1}^* = 2^{1+\gamma}\Gamma(2\gamma)(D\gamma)^{2\gamma}\left(\frac{\ln n}{n^2}\right)^{2\gamma} + o\left(\left(\frac{\ln n}{n^2}\right)^{2\gamma}\right),\ n\to\infty;$

2) if n is even, then $\varepsilon_{n,1} \sim \varepsilon_{n,1}^*,\ n\to\infty.$

Besides, for the majorant of uniform approximation the following asymptotic equality holds

$$\varepsilon_n^*\left(A_{n,1}\right) \sim \frac{2^{1-\gamma}\Gamma(2\gamma)\beta^{2\gamma}}{n^{2\gamma}} + \Phi\left(A_{n,1}, n\right),\ \beta \in (0, 1],\ n \to \infty,$$

where

$$\Phi\left(A_{n,1}, n\right) = \begin{cases} \dfrac{\sqrt{1+\beta^2}\,\Gamma(1-\gamma)\left(1-\beta^2\right)^{1-\gamma}}{\sqrt{2}(1+\beta)(2n\beta)^{1-\gamma}}\left(\dfrac{1-\beta}{1+\beta}\right)^n, & D = 1, \\ \dfrac{\sqrt{1+\beta^2}\,D^{2\gamma}\left(1-D^2\right)^{1-2\gamma}\left(D^2-\beta^2\right)}{2^{1-\gamma}\sqrt{1-D^4}\,(\beta+D)n\beta}\left(\dfrac{D-\beta}{D+\beta}\right)^n, & D \neq 1. \end{cases}$$

Remark 5. It is interesting to compare the asymptotic estimate of uniform approximation found in corollary 3 with the corresponding estimate in the polynomial case (29). We see that even in case of one fixed pole of the approximating function, the rate of uniform rational approximation has a much larger order of smallness, which reflects the peculiarities of rational approximation.

Approximation of some elementary functions

Many elementary functions can be represented as combinations of Markov functions. In this section, we consider an example of such a function and, as a consequence of theorem 6, we find the exact constant and the order of its approximations by the rational integral operator (4).

Consider the function $f(z) = (z-1)^\gamma,\ \gamma \in (0, +\infty)\backslash\mathbb{N}$. It is holomorphic in the region $\mathbb{C}\backslash(1, +\infty)$. The application of the Cauchy integral formula leads to the relation

$$(z-1)^\gamma = \frac{1}{2\pi i}\int_{\partial D}\frac{(\xi-1)^\gamma}{\xi - z}d\xi,\ z \in D,$$

where D is a circle of radius $a > 1$ centered at the origin and cut along a segment $[1, a]$. From the last formula it is easy to obtain (see [3; 4]), that the following equality holds for $|z| < a,\ z \,\overline{\in}\, (1, a)$,

$$(1-x)^\gamma = \hat{\mu}_1(x) + g(z), \tag{48}$$

where

$$\hat{\mu}_1(x) = -\frac{\sin\pi\gamma}{\pi}\int_1^a\frac{(t-1)^\gamma}{t-x}dt,\ g(z) = \frac{1}{2\pi i}\int_{|\xi|=a}\frac{(1-\xi)^\gamma}{\xi - z}d\xi.$$

The function $\hat{\mu}_1(x),\ x \in [0, 1]$, satisfies the conditions of theorem 3. Therefore,

$$\varepsilon_{n,q}\left(\hat{\mu}_1(x)\right) \sim \frac{2^{1+\gamma}}{\pi}|\sin\pi\gamma|\Gamma(2\gamma)\left(q^{2q+1}D\gamma^{2q-1}\right)^{2\gamma}\left(\prod_{k=1}^{q-1}(q-k)^2\right)^{2\gamma}\left(\frac{\ln^{2q-1}n}{n^{2q}}\right)^{2\gamma},\ n \to \infty.$$

The function $g(z)$ is holomorphic in the region D. According to the well-known result of S. N. Bernstein [25], the order of its polynomial approximation is exponential, that is, there are constants $M > 0,\ 0 < C < 1$, such that $\varepsilon_{n,q}\left(g(z),\ D\right) \leq MC^n$. Thus, the function $g(z),\ z \in D$, does not affect the order of approximations of the function $(1-x)^\gamma,\ x \in [0, 1]$. In other words,

$$\varepsilon_{n,q}\left((1-x)^\gamma\right) = \varepsilon_{n,q}\left(\hat{\mu}_1(x)\right) + o\left(\varepsilon_{n,q}\left(\hat{\mu}_1(x)\right)\right),\ n \to \infty.$$

From representation (48) and the last asymptotic equality, we obtain the following result.

Corollary 4 (approximation of the function $(1-x)^\gamma,\ \gamma > 0$). *For any fixed q and even n the relation holds*

$$\limsup_{n\to\infty}\left(\frac{n^{2q}}{\ln^{2q-1}n}\right)^{2\gamma}\varepsilon_{n,q}\left((1-x)^\gamma,\ [0, 1]\right) =$$

$$= \frac{2^{1+\gamma}}{\pi}|\sin\pi\gamma|\Gamma(2\gamma)\left(q^{2q+1}D\gamma^{2q-1}\right)^{2\gamma}\left(\prod_{k=1}^{q-1}(q-k)^2\right)^{2\gamma}. \quad (49)$$

Besides,

$$\lim_{n\to\infty} n^{2\gamma}\varepsilon_{n,0}\left((1-x)^{\gamma},\ [0,1]\right) = \frac{1}{\pi 2^{\gamma-1}}|\sin\pi\gamma|\Gamma(2\gamma),\ \gamma > 0.$$

It is known [26, p. 96] that the best uniform polynomial approximation of the considered function possesses the following property:

$$E_{2n}\left(|x|^{2\gamma};\ [-1,1]\right) = \frac{1}{2^{\gamma}}E_n\left((1-x)^{\gamma};\ [0,1]\right).$$

Using similar reasoning, after the necessary transformations from (49) we find that

$$\limsup_{n\to\infty}\left(\frac{n^{2q}}{\ln^{2q-1}n}\right)^{s}\varepsilon_{2n,2q}\left(|x|^{s},\ [-1,1]\right) = \frac{2}{\pi}\left|\sin\frac{\pi s}{2}\right|\Gamma(s)\left(q^{2q+1}D\left(\frac{s}{2}\right)^{2q-1}\right)^{s}\left(\prod_{k=1}^{q-1}(q-k)^2\right)^{s}. \quad (50)$$

Let $q = 1$ in the last relation. Then

$$\limsup_{n\to\infty}\left(\frac{n^{2}}{\ln n}\right)^{s}\varepsilon_{2n,2}\left(|x|^{s},\ [-1,1]\right) = \frac{2}{\pi}\left|\sin\frac{\pi s}{2}\right|\Gamma(s)\left(\frac{s}{2}\right)^{s}.$$

This result is contained in [27] in case of approximation by partial sums of Fourier series with respect to the system of Chebyshev – Markov of algebraic fractions. In particular, when $s = 1$ we obtain known equality, proved in [28],

$$\limsup_{n\to\infty}\frac{n^{2}}{\ln n}\varepsilon_{2n,2}\left(|x|,\ [-1,1]\right) = \frac{1}{\pi}.$$

Now we consider one more result, that follows from the formula (50). Substitution $x = \sin u$ leads to the asymptotic estimate

$$\limsup_{n\to\infty}\left(\frac{n^{2q}}{\ln^{2q-1}n}\right)^{s}\varepsilon_{2n,2q}\left(|\sin u|^{s},\ [0,\pi]\right) = \frac{2}{\pi}\left|\sin\frac{\pi s}{2}\right|\Gamma(s)\left(q^{2q+1}D\left(\frac{s}{2}\right)^{2q-1}\right)^{s}\left((q-1)!\right)^{2s}.$$

In this relation we put $s = 1$. Then we obtain known equality from [24]

$$\limsup_{n\to\infty}\frac{n^{2q}}{\ln^{2q-1}n}\varepsilon_{2n,2q}\left(|\sin u|,\ [0,\pi]\right) = \frac{q^{2q+1}}{2^{2q-2}\pi}\left((q-1)!\right)^{2}.$$

In particular, for $q = 1$ from the last formula we find

$$\limsup_{n\to\infty}\frac{n^{2}}{\ln n}\varepsilon_{2n,2}\left(|\sin u|,\ [0,\pi]\right) = \frac{1}{\pi}.$$

This asymptotic estimate coincides in order with the two-sided estimate contained in [29], obtained in case of approximation by partial sums of Fourier series with respect to the system of rational functions introduced independently by S. Takenaka [10] and F. Malmquist [11]. It should be noted, that, in comparison with work [29], here we found the exact constant.

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