О НЕКОТОРЫХ СВОЙСТВАХ РЕШЕТКИ ТОТАЛЬНО 
σ-ЛОКАЛЬНЫХ ФОРМАЦИЙ КОНЕЧНЫХ ГРУПП

И. Н. САФОНОВА 1), В. Г. САФОНОВ 1)

1) Белорусский государственный университет, пр. Независимости, 4, 220030, г. Минск, Беларусь

Все рассматриваемые в статье группы являются конечными. Пусть $\sigma = \{\sigma_i | i \in I\}$ — некоторое разбиение множества всех простых чисел $\mathbb{P}$. Если $n$ — целое число, $G$ — группа и $\mathfrak{F}$ — класс групп, то $\sigma(n) = \{\sigma_i | \pi(n) \neq \emptyset\}$,

$\sigma(G) = \sigma(|G|)$ и $\sigma(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \sigma(G)$. Функция $f : \sigma \rightarrow \{формации групп\}$ называется формационной $\sigma$-функцией. Для всякой формационной $\sigma$-функции $f$ класс $LF_\sigma(f)$ определяется следующим образом:

$$LF_\sigma(f) = \{G | G = 1 \text{ или } G \neq 1 \text{ и } G/O_{\sigma_i} \in f(\sigma_i) \text{ для всех } \sigma_i \in \sigma(G)\}.$$ 

Если для некоторой формационной $\sigma$-функции $f$ имеет место $\mathfrak{F} = LF_\sigma(f)$, то класс $\mathfrak{F}$ называют $\sigma$-локальным, а формационную $\sigma$-функцию $f$ — $\sigma$-локальным определением $\mathfrak{F}$. Всякую формацию считают 0-кратно $\sigma$-локальной.

АВТОРЫ:

Инна Николаевна Сафонова – кандидат физико-математических наук, доцент; заместитель декана по научной работе факультета прикладной математики и информатики.

Василий Григорьевич Сафонов — доктор физико-математических наук, профессор; проректор по научной работе, профессор кафедры высшей алгебры и защиты информации механико-математического факультета.

FOR CITATION:


https://doi.org/10.33581/2520-6508-2020-3-6-16
ON SOME PROPERTIES OF THE LATTICE OF TOTALLY $\sigma$-LOCAL FORMATIONS OF FINITE GROUPS

I. N. SAFONOVA, V. G. SAFONOVA

*Belarusian State University, 4 Niezaliežnasci Avenue, Minsk 220030, Belarus

Throughout this paper, all groups are finite. Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. If $n$ is an integer, $G$ is a group, and $F$ is a class of groups, then $\sigma(n) = \{\sigma_i | \pi(n)$ $\neq \emptyset\}$, $\sigma(G) = \sigma(\{G\})$, and $\sigma(F) = \bigcup_{G \in F} \sigma(G)$. A function $f$ of the form $f: \sigma \rightarrow \{\{\text{formations of groups}\}$ is called a formation $\sigma$-function. For any formation $\sigma$-function $f$, the class $LF_n(f)$ is defined as follows:

$$LF_n(f) = \{G \in 1$ or $G \neq 1$ and $G/O_{\sigma_n}(G) \in f(\sigma_i)$ for all $\sigma_i \in \sigma(G)\}$$

If for some formation $\sigma$-function $f$, we have $F = LF_n(f)$, then the class $F$ is called $\sigma$-local and $f$ is called a $\sigma$-local definition of $F$. Every formation is called $\sigma$-multiply $\sigma$-local. For $n \geq 0$, a formation $F$ is called $n$-multiply $\sigma$-local provided either $F = \{1\}$ is the class of all identity groups or $F = LF_n(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply $\sigma$-local for all $\sigma_i \in \sigma(F)$. A formation is called totally $\sigma$-local if it is $n$-multiply $\sigma$-local for all non-negative integer $n$. The aim of this paper is to study properties of the lattice of totally $\sigma$-local formations. In particular, we prove that the lattice of all totally $\sigma$-local formations is algebraic and distributive.

**Keywords:** finite group; formation $\sigma$-function; formation of finite groups; totally $\sigma$-local formation; lattice of formations.

Introduction

All groups under consideration are finite. The notations and definitions we use are borrowed from [1–3]. The basic properties and various applications of $\sigma$-local formations can be found in the articles [4–10].

A. Skiba presented [4] the concept of generalised locality or $\sigma$-locality of formations as a tool for studying the $\sigma$-properties of groups, i.e. properties depending on some partition $\sigma$ of the set of all primes. In [4], using $\sigma$-local formations, A. Skiba studied (weakly) $S^0_n$-closed and (weakly) $M^\sigma_n$-closed classes of finite groups. Some general properties of $\sigma$-local formations as well as their applications for studying $\Sigma^\sigma_n$-closed classes of meta-$\sigma$-nilpotent groups [5] and (weakly) $\Gamma^\sigma_n$-closed classes of finite groups [6], were obtained. Ch. Zhang and A. Skiba. Applications of the theory of $\sigma$-local formations were obtained by A. Skiba [7] for a lattice characterization of $\sigma$-soluble $PSO\Gamma$-groups, and also for constructing new sublattices of the lattice of all subgroups of the group generated by formation Fitting sets [10].

In [8; 9] Ch. Zhang, V. Safonov and A. Skiba described some general properties and examples of $n$-multiply $\sigma$-local formations and also consider one application of such formations in the theory of finite factorisable groups. In particular, in their paper [9] it was proved that the lattice of all $n$-multiply $\sigma$-local formations of finite groups is algebraic and modular.

A. Tsarev [11] proved that every law of the lattice of all formations is fulfilled in the lattice of all $n$-multiply $\sigma$-local formations of finite groups and that the lattice of all $n$-multiply $\sigma$-local formations of finite groups is modular but is not distributive for any non-negative integer $n$.

At the same time, the question on the algebraiciteness, modularity or distributivity of the lattice of all totally $\sigma$-local formations was an open problem. Note that the question on the distributivity or modularity of the lattice of all totally $\sigma$-local formations of finite groups was discussed by A. Tsarev in [11, question 3.2].
In this paper we will prove that the set \( l^0 \) of all totally \( \sigma \)-local formations of finite groups is a complete algebraic and distributive lattice. In the work, we study also some general properties of totally \( \sigma \)-local formations of finite groups.

We also note that the concept of generalised locality of formations was developed in papers [12; 13], where the main properties and some examples of Baer-\( \sigma \)-local formations were considered.

**Definitions and notations**

The basic definitions, notations and general properties of \( \sigma \)-local formations were discussed in the papers [4–10]. Recall some of the basic concepts of the theory of \( \sigma \)-local formations.

Let \( \sigma = \{ \sigma_i \mid i \in I \} \) be some partition of the set of all primes \( \mathbb{P} \). If \( n \) is an integer, \( G \) is a group, and \( \mathfrak{S} \) is a class of groups, then \( \sigma(n) = \{ \sigma_i \mid \sigma_i \cap n \neq \emptyset \} \), \( \sigma(G) = \sigma(\{G\}) \), and \( \sigma(\mathfrak{S}) = \bigcup_{G \in \mathfrak{S}} \sigma(G) \).

A group \( G \) is called [14]: \( \sigma \)-primary if \( G \) is a \( \sigma_i \)-group for some \( i \); \( \sigma \)-nilpotent if every chief factor \( H/K \) of \( G \) is \( \sigma \)-central in \( G \), that is, the semidirect product \( (H/K) \rtimes (G/C_G(H/K)) \) is \( \sigma \)-primary; \( \sigma \)-soluble if \( G = 1 \) or \( G \neq 1 \) and every chief factor of \( G \) is \( \sigma \)-primary.

We write \( \mathfrak{S}_n \) to denote the class of all \( \sigma \)-soluble groups and \( \mathfrak{N}_n \) to denote the class of all \( \sigma \)-nilpotent groups.

A class of groups \( \mathfrak{S} \) is called a formation if: (1) \( G/N \in \mathfrak{S} \) whenever \( G \in \mathfrak{S} \), and (2) \( G/N \cap R \in \mathfrak{S} \) whenever \( G/N \in \mathfrak{S} \) and \( G/R \in \mathfrak{S} \).

Any function \( f \) of the form \( f : \sigma \rightarrow \{ \text{formations of groups} \} \) is called a formation \( \sigma \)-function. For any formation \( \sigma \)-function \( f \) the class \( LF_{\sigma}(f) \) is defined as follows:

\[
LF_{\sigma}(f) = \left\{ G \mid G \text{ is a group} \right\} = \left\{ G,N \mid G/N \in \mathfrak{S} \right\}.
\]

If for some formation \( \sigma \)-function \( f \) we have \( \mathfrak{S} = LF_{\sigma}(f) \), then the class \( \mathfrak{S} \) is called \( \sigma \)-local and \( f \) is called \( \sigma \)-local definition of \( \mathfrak{S} \). We write \( F_{\sigma}(G) \) instead of \( O_{\sigma,\alpha}(G) = \sigma_{\sigma,\alpha}(G) \).

Every formation is called 0-multiply \( \sigma \)-local. For \( n > 0 \), the formation \( \mathfrak{S} \) is called \( n \)-multiply \( \sigma \)-local provided either \( \mathfrak{S} = \{1\} \) is the class of all identity groups or \( \mathfrak{S} = LF_{\sigma}(f) \), where \( f(\sigma) = (n-1) \)-multiply \( \sigma \)-local for all \( \sigma \in \mathfrak{S} \). A formation is called totally \( \sigma \)-local if it is \( n \)-multiply \( \sigma \)-local for all non-negative integer \( n \).

The symbol \( l^0 \) denotes the set of all totally \( \sigma \)-local formations. Formations from \( l^0 \) are called \( l^0 \)-formations.

For any collection of groups \( \mathfrak{X} \), \( l^0 \)-form \( x \) denotes the totally \( \sigma \)-local formation generated by \( \mathfrak{X} \), i.e. \( l^0 \)-form \( x \) is the intersection of all totally \( \sigma \)-local formations containing the collection of groups \( \mathfrak{X} \). If \( \mathfrak{X} = \{ G \} \) for some group \( G \), then \( \mathfrak{S} = l^0 \)-form \( G \) is called a one-generated totally \( \sigma \)-local formation. For any two classes of groups \( \mathfrak{M} \) and \( \mathfrak{J} \) we put \( \mathfrak{M} \triangledown \mathfrak{J} = l^0 \)-form \( (\mathfrak{M} \cup \mathfrak{J}) \).

If \( f \) is a formation \( \sigma \)-function, then the symbol \( \text{Supp}(f) \) denotes the support of \( f \), that is, the set of all \( \sigma_i \) such that \( f(\sigma_i) \neq \emptyset \). A formation \( \sigma \)-function \( f \) is called \( l^0 \)-valued if \( f(\sigma_i) \) is a totally \( \sigma \)-local formation for every \( \sigma_i \in \text{Supp}(f) \); integrated if \( f(\sigma_i) \subseteq LF_{\sigma}(f) \) for all \( i \).

If \( m \) and \( h \) are \( l^0 \)-valued formation \( \sigma \)-functions, then \( m \triangledown h \) is a formation \( \sigma \)-function such that \( m \triangledown h(\sigma_i) = m(\sigma_i) \triangledown h(\sigma_i) \) for all \( i \); we use also \( m \cap h \) to denote the formation \( \sigma \)-function such that \( m \cap h(\sigma_i) = m(\sigma_i) \cap h(\sigma_i) \) for all \( i \).

Every sequence \( \sigma_1, \sigma_2, \ldots, \sigma_n \) from \( \sigma \) is called a \( \sigma \)-sequence. For any \( \sigma \)-sequence \( \sigma_1, \sigma_2, \ldots, \sigma_n \), and for any collection of groups \( \mathfrak{X} \), the class of groups \( \mathfrak{X}(\sigma_1, \sigma_2, \ldots, \sigma_n) \) is defined recursively in the following way:

(1) \( \mathfrak{X}(\sigma_i) = \left\{ G/F_{\alpha_i}(G) \mid G \in \mathfrak{X} \right\} \); (2) \( \mathfrak{X}(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) = \left\{ G/F_{\alpha_1}(G) \mid G \in \mathfrak{X}(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \right\} \).

For any \( l^0 \)-formation \( \mathfrak{S} \), we set \( \mathfrak{S}_{l^0}(\sigma_i) = \mathfrak{S}_{l^0} \)-form \( (\mathfrak{S}(\sigma_i)) \), if \( \sigma_i \in \sigma(\mathfrak{S}) \), and \( \mathfrak{S}_{l^0}(\sigma_i) = \emptyset \), if \( \sigma_i \not\in \sigma(\mathfrak{S}) \).

If \( \mathfrak{S} \in l^0 \), then the symbol \( \mathfrak{S}_{l^0} \) denotes the smallest \( l^0 \)-valued formation of \( \mathfrak{S} \), i.e. \( \mathfrak{S}_{l^0} = \bigcap_{\sigma \in \mathfrak{S}} \mathfrak{S}_\sigma \), where \( \{ f_j \mid j \in J \} \) is the set of all \( l^0 \)-valued definitions of \( \mathfrak{S} \).

We say that a \( \sigma \)-sequence \( \sigma_1, \sigma_2, \ldots, \sigma_n \) is suitable for \( \mathfrak{S} \) (or \( \mathfrak{S} \)-suitable), if \( \sigma_i \in \sigma(\mathfrak{S}) \) and for any \( j \in \{2, \ldots, n\} \) we have \( \sigma_j \in \sigma(\mathfrak{S}(\sigma_1, \sigma_2, \ldots, \sigma_{j-1})) \).
Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an $\mathfrak{F}$-suitable $\sigma$-sequence. Then the $l^\sigma_\infty$-valued $\sigma$-function $\mathfrak{F}^\sigma_\infty \alpha_1 \ldots \alpha_n$ is defined recursively as follows: (1) $\mathfrak{F}^\sigma_\infty \alpha_1 = (\mathfrak{F}^\sigma_\infty (\alpha_1))^\sigma$; (2) $\mathfrak{F}^\sigma_\infty \alpha_1 \ldots \alpha_n = (\mathfrak{F}^\sigma_\infty \alpha_1 \ldots \alpha_{n-1} (\alpha_n))^\sigma$.

For any group $G$ and a non-empty formation $\mathfrak{F}$ by $G^\mathfrak{F}$ denote the $\mathfrak{F}$-residual of $G$, i.e. the intersection of all subgroups $N$ of $G$ such that $G/N \in \mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{G}$ are formations, then $\mathfrak{F} \mathfrak{G} = \{G \mid G^\mathfrak{G} \in \mathfrak{F}\}$ is called the Gaschütz product of formations $\mathfrak{F}$ and $\mathfrak{G}$.

**Auxiliary results**

We need some well-known results, which we present in the form of the following lemmas.

**Lemma 1** [9]. If the class of groups $\mathfrak{F}_j$ is an n-multiply $\sigma$-local formation for all $j \in J$, then the class $\cap_{j \in J} \mathfrak{F}_j$ is also n-multiply $\sigma$-local formation.

Recall that if $f$ is a formation $\sigma$-function, then the symbol $\text{Supp}(f)$ denotes the support of $f$, that is, the set of all $\sigma_i$ such that $f(\sigma_i) \neq \emptyset$.

**Lemma 2** [5; 9]. Let $f$ and $h$ be formation $\sigma$-functions and let $\Pi = \text{Supp}(f)$. Suppose that $\mathfrak{F} = LF(\sigma)(f) = LF(\sigma)(h)$.

1. $\Pi = \sigma(\mathfrak{F})$.
2. $\mathfrak{F} = \left( \cap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i} f(\sigma_i) \right) \cap \mathfrak{G}_\Pi$. Hence $\mathfrak{F}$ is a saturated formation.
3. If every group in $\mathfrak{F}$ is $\sigma$-soluble, then $\mathfrak{F} = \left( \cap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i} f(\sigma_i) \right) \cap \mathfrak{G}_\Pi$.
4. If $\sigma_i \in \Pi$, then $\mathfrak{G}_{\sigma_i} (f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i} (h(h) \cap \mathfrak{F}) \subseteq \mathfrak{F}$.
5. $\mathfrak{F} = LF(\sigma)(F)$, where $F$ is the unique formation $\sigma$-function such that $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ for all $\sigma_i \in \Pi$ and $F(\sigma_i) = \emptyset$ for all $\sigma_i \in \Pi'$. Moreover, $F(\sigma_i) = \mathfrak{G}_{\sigma_i} (f(\sigma_i) \cap \mathfrak{F})$ for all $i$.

**Lemma 3** [9]. If $\mathfrak{F}$ is a non-empty formation and $f(\sigma_i) = \mathfrak{F}$ for all $i$, then $LF(\sigma)(f) = \mathfrak{G}_{\sigma_i} \mathfrak{F}$.

**Lemma 4** [9]. If $\mathfrak{F} = \cap_{j \in J} \mathfrak{F}_j$ and $\mathfrak{G}_j = LF(\sigma)(f_j)$ for all $j \in J$, then $\mathfrak{F} = LF(\sigma)(f)$, where $f(\sigma_i) = \cap_{j \in J} f_j(\sigma_i)$ for all $\sigma_i \in \sigma(\mathfrak{F}) = \cap_{j \in J} \sigma(\mathfrak{F}_j)$ and $f(\sigma_i) = \emptyset$ for all $\sigma_i \in \sigma(\mathfrak{F})$. Moreover, if $f_j$ is integrated for all $j \in J$, then $f$ is also integrated.

**Lemma 5** [2, p. 41]. Let $A$ be a monolithic group and let $\text{Soc}(A)$ be a non-abelian group. Let $\mathfrak{M}$ be some homomorph. If $A \leq l_n \mathfrak{M}$, then $A \leq \mathfrak{M}$.

**Lemma 6** [2, p. 152]. Let $G$ be a group such that $O_p(G) = 1$, let $N_1 \times \ldots \times N_k = \text{Soc}(G)$, where $N_i$ is a minimal normal subgroup of $G$ (k ≥ 2). Let $M_i$ denote a maximal normal subgroup of $G$, which contains $N_i \times \ldots \times N_i \cap N_i \times \ldots \times N_i$ and does not contain $N_i$, $i \in \{1, \ldots, k\}$. Then
(a) the group $G/M_i$ is a monolithic and $\text{Soc}(G/M_i) = N_i M_i / N_i$ for any $i \in \{1, \ldots, k\}$;
(b) $N_i M_i / N_i$ is $G$-isomorphic to $N_i$;
(c) $O_p(G/M_i) = 1$;
(d) $M_i \cap \ldots \cap M_k = 1$.

The main results

Let $\mathfrak{X}$ be some collection of groups, $\sigma_i \in \sigma(\mathfrak{X})$, then the class of groups $\mathfrak{X}(\sigma_i)$ is defined as follows:

$$\mathfrak{X}(\sigma_i) = \left\{G \mid \mathfrak{F}_{\sigma_i}(G) \cap \mathfrak{G} \subseteq \mathfrak{F} \right\}$$

**Lemma 7**. Let $\mathfrak{F} = l^\sigma_\infty \text{form}(\mathfrak{X}) = LF(\sigma)(f)$ be the totally $\sigma$-local formation generated by $\mathfrak{X}$, where $f$ is an $\mathfrak{F}_{\sigma_i}(f)$-valued definition of $\mathfrak{F}$, and let $\Pi = \sigma(\mathfrak{X})$. Let $h$ be the formation $\sigma$-function such that $h(\sigma_i) = l^\sigma_\infty \text{form}(\mathfrak{X}(\sigma_i))$ for all $\sigma_i \in \Pi$ and $h(\sigma_i) = \emptyset$ for all $\sigma_i \in \Pi'$. Then

1. $\Pi = \sigma(\mathfrak{F})$;
2. $h$ is an $l^\sigma_\infty$-valued definition of $\mathfrak{F}$;
3. $h(\sigma_i) \subseteq f(\sigma_i) \cap \mathfrak{F}$ and for all $i$.

**Proof.** Since $\mathfrak{X} \subseteq \mathfrak{F}$ we have $\Pi \subseteq \sigma(\mathfrak{F})$. In view of [9, remark 2.4 (ii)], the class of all $\Pi$-groups $\mathfrak{G}_{\Pi}$ is a totally $\sigma$-local formation. Hence $\mathfrak{F} \subseteq \mathfrak{G}_{\Pi}$. Therefore, $\sigma(\mathfrak{F}) \subseteq \Pi$ and statement (1) holds.
Let $\mathfrak{H} = LF_\sigma(h)$. Then it is clear that $\mathfrak{X} \subseteq \mathfrak{H}$. On the other hand, since $h$ is an $l_\omega^\sigma$-valued, which implies that $\mathfrak{H}$ is a totally $\sigma$-local formation. Therefore, $\mathfrak{X} \subseteq \mathfrak{H}$.

Since $\mathfrak{X}(\sigma_i) \subseteq f(\sigma_j)$ and the formation $f(\sigma_i)$ is totally $\sigma$-local we have $h(\sigma_i) \subseteq f(\sigma_j)$ for all $\sigma_i \in \sigma$. Therefore, $\mathfrak{H} \subseteq \mathfrak{X}$. Hence $\mathfrak{H}$ and $\mathfrak{X}$ so statements (2) and (3) hold. The lemma is proved.

**Lemma 8.** Let $\mathfrak{L}_j = LF_\sigma(f_j)$ be a totally $\sigma$-local formation, where $f_j$ is the smallest $l_\omega^\sigma$-valued definition of $\mathfrak{L}_j, j \in J$. Then $\nu^\alpha(\mathfrak{L}_j, f_j)$ is the smallest $l_\omega^\sigma$-valued definition of $\mathfrak{L} = \mathfrak{L}_j, j \in J$.

**Proof.** Let $\mathfrak{L}$ be the smallest $l_\omega^\sigma$-valued definition of $\mathfrak{L}_j, f = \nu^\alpha(\mathfrak{L}_j, f_j)$, and $\Pi = \sigma(\cup_{j \in J} \mathfrak{L}_j) = \cup_{j \in J} \sigma(\mathfrak{L}_j)$. Then $\sigma(\mathfrak{L}) = \Pi$ by lemma 7 (1). Now we show that $l(\sigma_j) = f(\sigma_j)$ for all $\sigma_j \in \sigma$.

Let $\sigma_j \in \sigma(\mathfrak{L})$. Then for any $j \in J$ we have $f(\sigma_j) = \mathfrak{L}_j$. Hence $f(\sigma_j) = \mathfrak{L}$. Similary, in view of lemma 7, $l(\sigma_j) = \mathfrak{L}$. Therefore, $l(\sigma_j) = f(\sigma_j)$.

Now suppose that $\sigma_j \in \Pi$. Then there exists $j_i \in J$ such that $\sigma_j \in \sigma(\mathfrak{L}_{j_i})$. From lemma 7 it follows that $f_{j_i}(\sigma_j) \neq \emptyset$ and

$$l(\sigma_j) = l_{j_i}^\sigma \text{ form } (G/F_{j_i}(\sigma_i) \big| G \in \cup_{j \in J} \mathfrak{L}_j) = l_{j_i}^\sigma \text{ form } (\cup_{j \in J} l_{j_i}^\sigma \text{ form } (G/F_{\sigma_i}(\sigma_i) \big| G \in \mathfrak{L}_j) =$$

$$= l_{j_i}^\sigma \text{ form } (\cup_{j \in J} f_{\sigma_i}(\sigma_j) \big| j \in J) = (\nu^\alpha(\mathfrak{L}_j, f_j)) (\sigma_j) = f(\sigma_j).$$

Therefore, $l(\sigma_j) = f(\sigma_j)$ for all $\sigma_j \in \Pi$. Thus, $l = f$. The lemma is proved.

**Lemma 9.** Let $\mathfrak{H}_j = LF_\sigma(h_j)$, where $h_j$ is integrated $l_\omega^\sigma$-valued definition of $\mathfrak{H}_j, j = 1, 2$. Then $\mathfrak{H} = \mathfrak{H}_1 \cup^\sigma \mathfrak{H}_2 = LF_\sigma(h)$, where $h = h_1 \cup^\sigma h_2$ is integrated.

**Proof.** Let $l_j$ be the smallest $l_\omega^\sigma$-valued definition of $\mathfrak{H}_j$ and let $H_j$ be the canonical $\sigma$-local definition of $\mathfrak{H}_j, j = 1, 2$. In view of lemmas 2 (5) and 7 we have $l_j(\sigma) \subseteq h_j(\sigma) \subseteq H_j(\sigma)$ for all $\sigma_j$. Besides, lemmas 2 (5) and 7 imply also that

$$l(\sigma_j) = l_{\sigma_j}^\sigma \text{ form } (h_{\sigma_j} \cup \mathfrak{H}_2(\sigma_j)) \subseteq l_{\sigma_j}^\sigma \text{ form } (\mathfrak{H}_1(\sigma_j) \cup \mathfrak{H}_2(\sigma_j)) = l_{\sigma_j}^\sigma \text{ form } (l_{\sigma_j}^\sigma \sigma_j \cup l_{\sigma_j}^\sigma (\sigma_j)) \subseteq$$

$$\subseteq l_{\sigma_j}^\sigma \text{ form } (h(\sigma_j) \cup h(\sigma_j)) = h(\sigma_j) \subseteq \sigma(\mathfrak{H}_j) = H(\sigma_j).$$

Hence $l(\sigma_j) \subseteq h(\sigma_j) \subseteq H(\sigma_j)$ for all $\sigma_j$. Therefore, $\mathfrak{H} = LF_\sigma(h)$. The lemma is proved.

**Lemma 10.** Let $\mathfrak{K}$ be a non-empty formation. Then the formation $\sigma(\mathfrak{K})$ is totally $\sigma$-local.

**Proof.** Let $\mathfrak{M} = \sigma(\mathfrak{K})$. By lemma 3 the formation $\mathfrak{H} = \sigma(\mathfrak{K})$ is $\sigma$-local and $\mathfrak{H} = LF_\sigma(h)$, where $h(\sigma_i) = \mathfrak{M}$ for all $\sigma_i \in \sigma$. Since the Gaschütz product of formations is associative,

$$\mathfrak{H} = \sigma(\mathfrak{K}) \subseteq \sigma(\sigma(\mathfrak{K})) = \sigma(\mathfrak{K}) = \mathfrak{K}.$$

Therefore $\mathfrak{M}$ is a $\sigma$-local formation. On the other hand, $h(\sigma_i) = \mathfrak{M}$ for all $\sigma_i \in \sigma$. Hence $\mathfrak{M} = \sigma(\mathfrak{K}) = LF_\sigma(h)$ is 2-multiply $\sigma$-local. Therefore, the formation $\mathfrak{M}$ is $n$-multiply $\sigma$-local for any positive integer $n$. Consequently $\mathfrak{M}$ is totally $\sigma$-local. The lemma is proved.

**Lemma 11.** Let $\mathfrak{K} = LF_\sigma(f)$ and $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{K}$ for some $\sigma_i \in \sigma(G)$, then $G \in \mathfrak{K}$.

**Proof.** Since $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{K}$, we have $f(\sigma_i) \neq \emptyset$. But then $\sigma_i \in \sigma(\mathfrak{K})$ by lemma 2 (1). Moreover, $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{K}$ imply also that $G/O_{\sigma_i}(G) \subseteq O_{\sigma_i}(G) \in \sigma(\mathfrak{K})$. Hence $G \in \sigma(\mathfrak{K}) = f(\sigma_i) \cap \mathfrak{K}$. In view of lemma 2 (4) we have $G \in \sigma(\mathfrak{K}) = f(\sigma_i) \cap \mathfrak{K}$, where $\sigma(\mathfrak{K})$ is totally $\sigma$-local. The lemma is proved.

**Lemma 12.** Let $\mathfrak{L}_j = LF_\sigma(f_j)$, where $f_j$ is an integrated $l_\omega^\sigma$-valued definition of $\mathfrak{L}_j, j \in J$. If $\mathfrak{L} = \cap_{j \in J} \mathfrak{L}_j$ is a totally $\sigma$-local formation for all positive integer $n$. Then in view of lemma 1, $\cap_{j \in J} \mathfrak{L}_j$ is an $n$-multiply $\sigma$-local formation for all positive integer $n$. Therefore $\cap_{j \in J} \mathfrak{L}_j$ is totally $\sigma$-local. Besides, since $f = \cap_{j \in J} f_j$ and $f_j$ is an $l_\omega^\sigma$-valued definition of $\mathfrak{L}_j$, we see that $f$ is an $l_\omega^\sigma$-valued $\sigma$-function.
Let $\mathfrak{F} = LF_\sigma(f)$. Now we show that $\mathfrak{F} = \mathfrak{F}_1$. First assume that $G \in \mathfrak{F}$ and let $\sigma_i \in \sigma(G)$. Then $G/F_{\{\sigma_i\}}(G) \in F_j(\sigma_j)$, hence $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_j)$ for all $j \in J$. Therefore for all $\sigma_i \in \sigma(G)$ we obtain $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_j)$, but then $G \in \mathfrak{F}_j$ for all $j \in J$ and consequently, $G \in \cap_{j \in J} \mathfrak{F}_j = \mathfrak{F}$. Thus $\mathfrak{F} \subseteq \mathfrak{F}_1$.

Now suppose that $G \in \mathfrak{F} = \cap_{j \in J} \mathfrak{F}_j$ and let $\sigma_i \in \sigma(G)$. Then $G \in \mathfrak{F}_j$ for all $j \in J$. Therefore, $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_j)$ for all $j \in J$. Hence $G/F_{\{\sigma_i\}}(G) \in \cap_{j \in J} f_j(\sigma_j) = f(\sigma_j)$ for all $\sigma_i \in \sigma(G)$. Consequently, $G \in \mathfrak{F}_1$ and $\mathfrak{F} \subseteq \mathfrak{F}_1$.

Finally, since $f_j$ is integrated for all $j \in J$, we have $f(\sigma) = \cap_{j \in J} f_j(\sigma) \subseteq f_j(\sigma_j) \subseteq \mathfrak{F}_j$ for all $\sigma_i \in \sigma$. Hence $f(\sigma) \subseteq \cap_{j \in J} \mathfrak{F}_j = \mathfrak{F}$ and $f$ is an integrated $l_{\sigma}$-valued definition of $\mathfrak{F}$. The lemma is proved.

**Algebraicity of the lattice of all totally $\sigma$-local formations**

**Theorem 1.** The set $l_{\sigma}$ of all totally $\sigma$-local formations is a complete algebraic lattice in which, for any set $\{\mathfrak{F}_j \mid j \in J\} \subseteq l_{\sigma}$, the intersection $\cap_{j \in J} \mathfrak{F}_j$ is the greatest lower bound and $l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$ is the smallest upper bound of $\{\mathfrak{F}_j \mid j \in J\}$ in $l_{\sigma}$.

**Proof.** It is clear that the set $l_{\sigma}$ is partially ordered with respect to set inclusion. Since by [9, remark 2.4 (ii)], the formation of all groups $\mathfrak{F}$ is totally $\sigma$-local we have $\mathfrak{F}$ is the largest element in $l_{\sigma}$. It follows from lemma 12 that $\cap_{j \in J} \mathfrak{F}_j \in l_{\sigma}$. Therefore, $\cap_{j \in J} \mathfrak{F}_j$ is the greatest lower bound of $\{\mathfrak{F}_j \mid j \in J\}$ in $l_{\sigma}$, which implies that $l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$ is the smallest upper bound of $\{\mathfrak{F}_j \mid j \in J\}$ in $l_{\sigma}$.

Now we show that for every group $A$ the one-generated totally $\sigma$-local formation $l_{\sigma}$ form $(A)$ is a compact element in $l_{\sigma}$. Let $A$ be a counterexample minimal order and

$$\mathfrak{F} = l_{\sigma}$ form $(A) \subseteq l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$, $$

where $\mathfrak{F}_j$ is a totally $\sigma$-local formation, $j \in J$. If $A$ is a $\sigma$-group for some $i$, then $\mathfrak{F} = \mathfrak{F}_{\sigma_i}$. Since $\mathfrak{F} \subseteq l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$, we have $\sigma_i(\mathfrak{F}) \subseteq \sigma_i(\cup_{j \in J} \mathfrak{F}_j) = \cup_{j \in J} \sigma_i(\mathfrak{F}_j)$ by lemma 7. Therefore, there is $j_i$ such that $\sigma_i \in \sigma_i(\mathfrak{F}_{j_i})$. But then $\mathfrak{F} = \sigma_i \subseteq \mathfrak{F}_{j_i}$, by lemma 2. This contradiction shows that $A$ is not $\sigma$-primary.

Now we show that $A$ is monolithic. Suppose that it is false and let $N_1, N_2$ be minimal normal subgroups of $A$, where $N_1 \neq N_2$. Let $L = l_{\sigma}$ form $(A/N_1)$, $M = l_{\sigma}$ form $(A/N_2)$. It is clear that $\mathfrak{F} = L \vee m \mathfrak{M}$. By inductive hypothesis for groups $A/N_1$ and $A/N_2$ our statement is true. Then since

$$L = l_{\sigma}$ form $(A/N_1) \subseteq \mathfrak{F} \subseteq l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$, $$

$$M = l_{\sigma}$ form $(A/N_2) \subseteq \mathfrak{F} \subseteq l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$,

there are $j_1, \ldots, j_k$ and $s_1, \ldots, s_n$ such that

$$L \subseteq l_{\sigma}$ form $(\mathfrak{F}_{j_1} \cup \ldots \cup \mathfrak{F}_{j_k})$$

and

$$M \subseteq l_{\sigma}$ form $(\mathfrak{F}_{s_1} \cup \ldots \cup \mathfrak{F}_{s_n})$$,

But then we have

$$\mathfrak{F} = L \vee m \mathfrak{M} \subseteq l_{\sigma}$ form $(\mathfrak{F}_{j_1} \cup \ldots \cup \mathfrak{F}_{j_k} \cup \mathfrak{F}_{s_1} \cup \ldots \cup \mathfrak{F}_{s_n})$$,

a contradiction. Hence $A$ is a monolithic group.

Let $P = \text{Soc}(A)$. Assume that $P$ is not a $\sigma$-primary group. Since $A \in l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$ we have $A \in \sigma_{\mathfrak{F}_{\sigma_i}}(\cup_{j \in J} \mathfrak{F}_j)$ by lemma 10. But $P$ is not a $\sigma$-primary group, therefore, $A \in \text{form}((\cup_{j \in J} \mathfrak{F}_j))$. Using lemma 5, we obtain $A \in \cup_{j \in J} \mathfrak{F}_j$. Hence there is $j_m \in J$ such that $A \in \mathfrak{F}_{j_m}$. This contradiction shows that $P$ is a $\sigma$-group for some $i$. Therefore, $F_{\sigma_i}(A) = O_{\sigma_i}(A)$.

Let $f_j, f, h$ are smallest $l_{\sigma}$-valued definitions of formations $\mathfrak{F}_j$, $\mathfrak{F}$ and $\mathfrak{F} = l_{\sigma}$ form $(\cup_{j \in J} \mathfrak{F}_j)$, respectively. In view of lemma 8 we have $h = \vee m (f_j \mid j \in J)$. Since $O_{\sigma_i}(A) = F_{\sigma_i}(A)$ and $A \in \mathfrak{F}$, we have

$$A/O_{\sigma_i}(A) = A/F_{\sigma_i}(A) \subseteq h(\sigma_j) = \vee m (f_j(\sigma_j) \mid j \in J)$$.

Since $|A/O_{\sigma_i}(A)| < |A|$, by our inductive hypothesis there are $j_1, \ldots, j_r \in J$ such that
\[ l^\sigma \text{ form } \left( A \setminus O_{\sigma}(A) \right) \subseteq \left( f_{j_1}(\sigma_1) \vee \ldots \vee f_{j_1}(\sigma_1) \right). \]

By lemma 8, \( l = f_{j_1} \vee \ldots \vee f_{j_1} \) is the smallest \( l_\sigma \)-valued definition of \( \mathcal{L} = \mathcal{F}_1 \vee \ldots \vee \mathcal{F}_j \). But then \( A \setminus O_{\sigma}(A) \in l(\sigma_1) \vee \ldots \vee f_{j_1}(\sigma_1) \). Since \( l \) is integrated, \( A \in \mathcal{L} \) by lemma 11.

Thus, \( l = l_\sigma \) form \( \mathcal{F}_j \subseteq \mathcal{L} = \mathcal{F}_1 \vee \ldots \vee \mathcal{F}_j \). This contradiction shows that every one-generated totally \( \sigma \)-local formation \( l_\sigma \) form \( A \) is a compact element in \( l_\sigma \).

It is clear that for any totally \( \sigma \)-local formation \( \mathcal{F} \) we have \( \mathcal{F} = l_\sigma \) form \( \left( \vee_{i \in T} \mathcal{F}_i \right) \), where \( \{ \mathcal{F}_i \} \subseteq T \) is the set of all one-generated totally \( \sigma \)-local formations contained in \( \mathcal{F} \). Hence the lattice \( l_\sigma \) is algebraic. The theorem is proved.

In the classical case, when \( \sigma = \sigma^1 = \{ \{2\}, \{3\}, \ldots \} \) we get from theorem 1 the following known results.

**Corollary 1** [15]. The lattice \( l_\sigma \) of all totally local formations is algebraic.

**Corollary 2** [2]. The lattice of all soluble totally local formations is algebraic.

**Distributivity of the lattice of all totally \( \sigma \)-local formations.**

Recall that if \( \mathcal{F} \) is a totally \( \sigma \)-local formation, then \( \mathcal{F}^\sigma \) denotes the smallest \( l_\sigma \)-valued definition of \( \mathcal{F} \). If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) is an \( \mathcal{F} \)-suitable \( \sigma \)-sequence, then the \( l_\sigma \)-valued \( \sigma \)-function \( \mathcal{F}^\sigma \) is defined recursively as follows: (1) \( \mathcal{F}^\sigma = \mathcal{F} \); (2) \( \mathcal{F}^\sigma = \mathcal{F} \).

Let \( \mathcal{F}, \mathcal{M}, \) and \( \mathcal{X} \) be totally \( \sigma \)-local formations. Let \( \alpha_1, \ldots, \alpha_n \) be some suitable \( \sigma \)-sequence for \( \mathcal{F}, \mathcal{M}, \) and \( \mathcal{X} \). Then by \( \mathcal{L}^\sigma, \mathcal{F}^\sigma, \mathcal{L}^\sigma \alpha_1, \mathcal{L}^\sigma \alpha_1, \ldots, \mathcal{F}^\sigma \alpha_1 \ldots \alpha_n \) we denote formation \( \sigma \)-functions such that \( \mathcal{L}^\sigma = \mathcal{X} \vee \mathcal{L}^\sigma \), \( \mathcal{F}^\sigma = \mathcal{X} \vee \mathcal{M} \), \( \mathcal{L}^\sigma \alpha_1 = \mathcal{X} \vee \mathcal{L}^\sigma \alpha_1 \), and \( \mathcal{F}^\sigma \alpha_1 = \mathcal{X} \vee \mathcal{M} \alpha_1 \).

**Lemma 13.** Let \( \mathcal{L} = (\mathcal{X} \vee \mathcal{M}) \cap \mathcal{N} \), \( \mathcal{F} = (\mathcal{X} \vee \mathcal{M}) \cap \mathcal{N} \), where \( \mathcal{M}, \mathcal{X}, \) and \( \mathcal{N} \) are totally \( \sigma \)-local formations. Then

1. \( \sigma(\mathcal{L}) = \sigma(\mathcal{N}) \);
2. If \( \alpha_1, \ldots, \alpha_n \) is a suitable \( \sigma \)-sequence for \( \mathcal{X}, \mathcal{M}, \) and \( \mathcal{N} \), then the formation \( \sigma \)-functions

\[ \mathcal{L}^\sigma, \mathcal{F}^\sigma, \mathcal{L}^\sigma \alpha_1, \mathcal{L}^\sigma \alpha_1, \ldots, \mathcal{L}^\sigma \alpha_1 \ldots \alpha_n \]

are integrated \( l_\sigma \)-valued definitions of the formations

\[ \mathcal{L}, \mathcal{F}, \mathcal{L}^\sigma (\alpha_1), \mathcal{F}^\sigma (\alpha_1), \ldots, \mathcal{L}^\sigma (\alpha_1 \ldots \alpha_n), \mathcal{F}^\sigma (\alpha_1 \ldots \alpha_n) \], respectively.

**Proof.** Let \( \mathcal{F}_1 = \mathcal{X} \vee \mathcal{M}, \mathcal{L}_1 = \mathcal{X} \cap \mathcal{N}, \mathcal{L}_2 = \mathcal{M} \cap \mathcal{N}, \mathcal{h}_1 = \mathcal{X} \vee \mathcal{M} \), \( l_1 = \mathcal{X} \cap \mathcal{N} \), and \( l_2 = \mathcal{M} \cap \mathcal{N} \).

By lemmas 8 and 12, it follows that \( \mathcal{F}_1 = \mathcal{L}^\sigma (h_1), \mathcal{L}_1 = \mathcal{L}^\sigma (l_1), \mathcal{L}_2 = \mathcal{L}^\sigma (l_2) \), and \( \mathcal{h}_1, \mathcal{h}_2 \) are integrated \( l_\sigma \)-valued \( \sigma \)-functions.

1. Since the inclusion \( \mathcal{L} \subseteq \mathcal{F} \) is obvious, we obtain \( \sigma(\mathcal{L}) \subseteq \sigma(\mathcal{F}) \). Let \( \sigma_1 \in \sigma(\mathcal{F}) \backslash \sigma(\mathcal{N}) \). Since \( \sigma_1 \in \sigma(\mathcal{F}_1) \), we have \( h_1(\mathcal{F}_1) = \emptyset \) by lemma 7. If \( \sigma_1 \notin \sigma(\mathcal{X}) \cup \sigma(\mathcal{M}) \), then it follows from lemma 7 that \( \mathcal{X} \sigma_1 = \emptyset \) and \( \mathcal{M} \sigma_1 = \emptyset \), a contradiction. Hence \( \sigma_1 \in \sigma(\mathcal{X}) \cup \sigma(\mathcal{M}) \).

2. Since \( \mathcal{L} = \mathcal{L}_1 \vee \mathcal{L}_2 \) and \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \), it follows from lemmas 9 and 12, that

\[ \mathcal{L}^\sigma = \mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{X} \vee \mathcal{M} \cap \mathcal{N}, \mathcal{F}^\sigma = \mathcal{X} \vee \mathcal{M} \cap \mathcal{N}, \mathcal{L}^\sigma \alpha_1 \ldots \alpha_n = \mathcal{X} \vee \mathcal{M} \alpha_1 \ldots \alpha_n \]

are integrated \( l_\sigma \)-valued definitions of the formations \( \mathcal{L} \) and \( \mathcal{F} \), respectively.
Let $\alpha_1, ..., \alpha_n$ be a suitable sequence for $\mathcal{X}$, $\mathcal{M}$, and $\mathfrak{F}$. Since by the definition $\mathcal{X}_n^\sigma \alpha_1 ... \alpha_j$, $\mathcal{M}_n^\sigma \alpha_1 ... \alpha_j$, $\mathfrak{F}_n^\sigma \alpha_1 ... \alpha_j$ are smallest $t_n^\sigma$-valued definitions of the formations

$$X_n^\sigma \alpha_1 ... \alpha_j \langle \alpha_j \rangle, M_n^\sigma \alpha_1 ... \alpha_j \langle \alpha_j \rangle,$$

respectively, it follows from lemmas 9 and 12 that

$$\mathfrak{F}_n^\sigma \alpha_1 ... \alpha_j \langle \alpha_j \rangle = (X_n^\sigma \alpha_1 ... \alpha_j \cap \mathfrak{F}_n^\sigma \alpha_1 ... \alpha_j \langle \alpha_j \rangle),$$

are integrated $t_n^\sigma$-valued definitions of the formations

$$\mathfrak{F}_n^\sigma \alpha_1 ... \alpha_j \langle \alpha_j \rangle$$

respectively. The lemma is proved.

**Lemma 14.** Let $\mathcal{M}$, $\mathcal{X}$, and $\mathfrak{F}$ be totally $\sigma$-local formations. Let $G$ be a monolithic group and $\text{Soc}(G)$ is not $\sigma$-primary. If $G \in \mathcal{F} \cap \left(X \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}\right)$, then $G \in \left(X \cap \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}\right)$.

**Proof.** Let $G \in \mathcal{F} \cap \left(X \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}\right)$. It follows from lemma 10 that $\mathcal{G}_\sigma$ form $\mathcal{X} \cup \mathcal{M}$ is a totally $\sigma$-local formation. Therefore, $t_n^\sigma$ form $\mathcal{X} \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}$ form $\mathcal{X} \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}$. Since $G$ is a monolithic group and $\text{Soc}(G)$ is not a $\sigma$-primary group, we have $G \in \mathcal{X} \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}$ and $\text{Soc}(G)$ is a non-abelian group. But then by lemma 5, it follows that $G \in \mathcal{X} \cup \mathcal{M}$. Since $G \in \mathcal{F}$, we obtain $G \in \left(X \cap \mathcal{F} \cup \left(M \cup \mathfrak{F}\right)\right)$. Hence,

$$G \in t_n^\sigma \left(\mathcal{X} \cap \mathcal{F} \cup \left(M \cup \mathfrak{F}\right)\right) = \left(X \cap \mathcal{F} \cup \left(M \cup \mathfrak{F}\right)\right) \cup \mathfrak{F}.$$

The lemma is proved.

**Theorem 2.** The lattice $t_n^\sigma$ of all totally $\sigma$-local formations is distributive.

**Proof.** Suppose that this formation is false. Then there exist totally $\sigma$-local formations $\mathcal{M}$, $\mathcal{X}$, and $\mathfrak{F}$ such that

$$\left(X \cap \mathfrak{F}\right) \cup \left(M \cup \mathfrak{F}\right) \neq \left(X \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}\right) \cap \mathfrak{F}.$$

Let $L = \left(X \cap \mathcal{F} \cup \left(M \cup \mathfrak{F}\right)\right)$ and $H = \left(X \cup \mathcal{F} \cup \mathcal{M} \cup \mathfrak{F}\right)$. Since the inclusion $L \subseteq H$ is obvious, we obtain $H \nsubseteq L$. Let $G$ be a group of minimal order in $H \setminus L$. In view that $L$ is a $\sigma$-local formation, we see that $G$ is a monolithic group with a unique minimal normal subgroup $P = G^\sigma$.

If $P$ is not $\sigma$-primary, then $G \in L$ by lemma 14. This contradiction shows that $P$ is a $\sigma$-group for some $\sigma \in \mathcal{S}(\mathfrak{F})$. It follows from lemma 13 that $\sigma \in \mathcal{S}(\mathcal{L})$, $L = LF_\sigma \left(M^\sigma\right)$, $H = LF_\sigma \left(M^\sigma\right)$, and $L^\sigma$, $H^\sigma$ are integrated $t_n^\sigma$-valued formations such that

$$L^\sigma(\sigma_1) = \left(X^\sigma(\sigma_1) \cap \mathfrak{F}^\sigma(\sigma_1)\right), H^\sigma(\sigma_1) = \left(X^\sigma(\sigma_1) \cup \mathfrak{F}^\sigma(\sigma_1)\right).$$

Since $L$ is a $\sigma$-local formation and $\sigma \in \mathcal{S}(\mathcal{L})$, we see that $G$ is not a $\sigma$-group and $L^\sigma(\sigma_1) \neq \emptyset$. On the other hand, since $P$ is a $\sigma$-group, $O_{\sigma_1}(G) = 1$ and $F_{\sigma_1}(G) = O_{\sigma_1}(G)$.

Since $L \subseteq H$ we have $L^\sigma(\sigma_1) \subseteq H^\sigma(\sigma_1)$ and since $G \in H \setminus L$, we claim that $L^\sigma(\sigma_1) \subset H^\sigma(\sigma_1)$. Indeed, by lemma 2, it follows that $G/F_{\sigma_1}(G) \in L^\sigma(\sigma_1)$. If $L^\sigma(\sigma_1) = H^\sigma(\sigma_1)$, then

$$G/O_{\sigma_1}(G) = G/F_{\sigma_1}(G) \in L^\sigma(\sigma_1) \subseteq H^\sigma(\sigma_1) = L^\sigma(\sigma_1)$$

and $G \in L$ by lemma 11. It is a contradiction. Therefore, $L^\sigma(\sigma_1) \subset H^\sigma(\sigma_1)$. Note also that the condition $L^\sigma(\sigma_1) \subset H^\sigma(\sigma_1)$ implies $X^\sigma(\sigma_1) \neq \emptyset$ and $M^\sigma(\sigma_1) \neq \emptyset$, since otherwise $L^\sigma(\sigma_1) = H^\sigma(\sigma_1)$. Hence $\sigma_1 \in \sigma(\mathcal{X}) \cap \sigma(\mathcal{M})$. Thus,

$$G_1 = G/F_{\sigma_1}(G) \in H^\sigma(\sigma_1) \setminus L^\sigma(\sigma_1), L^\sigma(\sigma_1) \neq \emptyset.$$

It follows from lemma 13 that

$$\sigma(L^\sigma(\sigma_1)) = \sigma(H^\sigma(\sigma_1)), L^\sigma(\sigma_1) = LF_\sigma \left(L^\sigma(\sigma_1)\right), H^\sigma(\sigma_1) = LF_\sigma \left(H^\sigma(\sigma_1)\right).$$
and \( L_\sigma G \), \( \tilde{L}_\sigma G \) are integrated \( L_\sigma G \)-valued definitions such that
\[
\tilde{L}_\sigma G = \left( X_\sigma G \cap \tilde{\tilde{\sigma}} G \right) \vee \left( M_\sigma G \cap \tilde{\tilde{\sigma}} G \right), \quad \tilde{L}_\sigma G = \left( X_\sigma G \vee M_\sigma G \cap \tilde{\tilde{\sigma}} G \right) \cap \tilde{\tilde{\sigma}} G.
\]

Since \( G_1 \not\subseteq \tilde{L}_\sigma G \), there exist \( \alpha_i \in \sigma G \) such that
\[
G_1 / F_{\{\alpha_i\}} \not\subseteq \tilde{L}_\sigma G \alpha_i.
\]

Note that since \( \alpha_i \in \sigma \left( \tilde{L}_\sigma G \alpha_i \right) \), we have \( \alpha_i \in \sigma \left( \tilde{L}_\sigma G \alpha_i \right) \) and \( \tilde{L}_\sigma G \alpha_i \neq \emptyset \). Obviously,
\[
\tilde{L}_\sigma G \alpha_i = \left( X_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) \vee \left( M_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) \subseteq \tilde{L}_\sigma G \alpha_i = \left( X_\sigma G \alpha_i \vee M_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right).
\]

Besides, since \( G_1 / F_{\{\alpha_i\}} \not\subseteq \tilde{L}_\sigma G \alpha_i \), we have \( \tilde{L}_\sigma G \alpha_i \subset \tilde{L}_\sigma G \alpha_i \). Therefore, \( \tilde{L}_\sigma G \alpha_i \neq \emptyset \) and \( M_\sigma G \alpha_i \neq \emptyset \). Hence \( \alpha_i \in \sigma \left( X_\sigma G \alpha_i \right) \cap \sigma \left( M_\sigma G \alpha_i \right) \).

Suppose that \( F_{\{\alpha_i\}} (G_1) = 1 \) and let \( N \) be a minimal normal subgroup of \( G_1 \). Then \( N \) is not \( \sigma \)-primary. If \( G_1 \) is a monolithic group, then since
\[
G_1 \in \tilde{L}_\sigma G \alpha_i = \left( X_\sigma G \alpha_i \vee M_\sigma G \alpha_i \right) \cap \tilde{\tilde{\sigma}} G \alpha_i,
\]
by lemma 14, it follows that
\[
G_1 \in \left( X_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) \vee \left( M_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) = \tilde{L}_\sigma G \alpha_i.
\]
This contradiction shows that the group \( G_1 \) is not monolithic.

Let \( \text{Soc} (G_1) = N_1 \times \ldots \times N_k \), where \( N_j \) is a minimal normal subgroup of \( G_1 \) and let \( M_j \) denote a maximal normal subgroup of \( G_1 \) such that \( M_j \) contains \( N_1 \times \ldots \times N_{j-1} \times N_{j+1} \times \ldots \times N_k \) and does not contain \( N_j \), \( j = 1, 2, \ldots, k \). By lemma 6, it follows that \( G_1 / M_j \) is a monolithic group with a non-\( \sigma \)-primary minimal normal subgroup \( N_j / M_j \) and \( N_j / M_j \) is \( G_1 \)-isomorphic to \( N_j \). Set \( B_j = G_1 / M_j \), \( j = 1, 2, \ldots, k \). Since
\[
B_j \in \tilde{L}_\sigma G \alpha_i = \left( X_\sigma G \alpha_i \vee M_\sigma G \alpha_i \right) \cap \tilde{\tilde{\sigma}} G \alpha_i,
\]
we have \( B_j \in \tilde{L}_\sigma G \alpha_i \) by lemma 14. It follows from lemma 6 (d) that \( G_1 \) is a subdirect product of \( B_1, \ldots, B_k \). Hence \( G_1 \in \tilde{L}_\sigma G \alpha_i \). This contradiction shows that \( F_{\{\alpha_i\}} (G_1) \neq 1 \).

On the other hand \( F_{\{\alpha_i\}} (G_1) \neq G_1 \), since otherwise \( G_1 / F_{\{\alpha_i\}} (G_1) = 1 \in \tilde{L}_\sigma G \alpha_i \neq \emptyset \). Thus,
\[
G_1 / F_{\{\alpha_i\}} (G_1) \in \tilde{L}_\sigma G \alpha_i \setminus \tilde{L}_\sigma G \alpha_i, \quad \tilde{L}_\sigma G \alpha_i \neq \emptyset, \quad 1 \neq F_{\{\alpha_i\}} (G_1) \subseteq G_1.
\]

Let \( G_2 = G_1 / F_{\{\alpha_i\}} (G_1) \). It follows from lemma 13 that
\[
\sigma \left( \tilde{L}_\sigma G \alpha_i \right) = \sigma \left( \tilde{\tilde{\sigma}} G \alpha_i \right),
\]
\[
\tilde{L}_\sigma G \alpha_i = LF_\sigma \left( \tilde{\tilde{\sigma}} G \alpha_i \right), \quad \tilde{\tilde{\sigma}} G \alpha_i = LF_\sigma \left( \tilde{\tilde{\sigma}} G \alpha_i \right),
\]
and \( \tilde{L}_\sigma G \alpha_i, \tilde{\tilde{\sigma}} G \alpha_i \) are integrated \( L_\sigma G \)-valued definitions such that
\[
\tilde{L}_\sigma G \alpha_i = \left( X_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) \vee \left( M_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right),
\]
\[
\tilde{\tilde{\sigma}} G \alpha_i = \left( X_\sigma G \alpha_i \vee M_\sigma G \alpha_i \cap \tilde{\tilde{\sigma}} G \alpha_i \right) \cap \tilde{\tilde{\sigma}} G \alpha_i.
\]
Since \( G_2 \neq \tilde{L}_\sigma G \alpha_i \), there exists \( \alpha_2 \in \sigma (G_2) \) such that
\[
G_2 / F_{\{\alpha_i\}} (G_2) \neq \tilde{L}_\sigma G \alpha_i \alpha_2.
\]
Hence,
\[ G_2/F_{\{\alpha_2\}}(G_2) \in \tilde{S}_2^\sigma, \sigma_1(\alpha_2) \setminus \tilde{S}_2^\sigma, \sigma_1(\alpha_2). \]

Considering \( G_2 \) in the same way as the group \( G_1 \), we obtain
\[ \alpha_2 \in \sigma(\tilde{X}^\sigma, \sigma, \alpha_1(\alpha_2)) \cap \sigma(\tilde{M}^\sigma, \sigma, \alpha_1(\alpha_2)), \]
\[ G_2/F_{\{\alpha_2\}}(G_2) \in \tilde{S}_2^\sigma, \sigma_1(\alpha_2) \setminus \tilde{S}_2^\sigma, \sigma_1(\alpha_2), \]

\[ \tilde{L}_2^\sigma, \sigma_1(\alpha_2) \neq \emptyset, \text{ and } 1 \neq F_{\{\alpha_2\}}(G_2) \subset G_2. \]

Put \( G_3 = G_2/F_{\{\alpha_2\}}(G_2) \). According to the same argument, we see that the group \( G_3 \) satisfies the analogous conditions: there exists
\[ \alpha_3 \in \sigma(\tilde{X}^\sigma, \sigma, \alpha_1(\alpha_2)) \cap \sigma(\tilde{M}^\sigma, \sigma, \alpha_1(\alpha_2)) \]

such that
\[ G_3/F_{\{\alpha_3\}}(G_3) \in \tilde{S}_3^\sigma, \sigma_2, \alpha_2(\alpha_3) \setminus \tilde{S}_3^\sigma, \sigma_2, \alpha_2(\alpha_3), \]

\[ \tilde{L}_3^\sigma, \sigma_1, \alpha_2(\alpha_3) \neq \emptyset, \text{ and } 1 \neq F_{\{\alpha_3\}}(G_3) \subset G_3. \]

Continuing this line of reasoning, we construct the groups
\[ G_4 = G_3/F_{\{\alpha_3\}}(G_3), \ldots, G_n = G_{n-1}/F_{\{\alpha_{n-1}\}}(G_{n-1}), \ldots \]

such that for any \( j \) the following conditions are satisfied:
\[ \alpha_{j-1} \in \sigma(\tilde{X}^\sigma, \sigma, \alpha_1 \ldots \alpha_{j-3}(\alpha_{j-2})) \cap \sigma(\tilde{M}^\sigma, \sigma, \alpha_1 \ldots \alpha_{j-3}(\alpha_{j-2})), \]
\[ G_j = G_{j-1}/F_{\{\alpha_{j-1}\}}(G_{j-1}) \in \tilde{S}_j^\sigma, \sigma_1, \alpha_1 \ldots \alpha_{j-2}(\alpha_{j-1}) \setminus \tilde{S}_j^\sigma, \sigma_1, \alpha_1 \ldots \alpha_{j-2}(\alpha_{j-1}), \]

\[ \tilde{L}_j^\sigma, \sigma_1, \alpha_1 \ldots \alpha_{j-2}(\alpha_{j-1}) \neq \emptyset, \text{ and } 1 \neq F_{\{\alpha_{j-1}\}}(G_{j-1}) \subset G_{j-1}. \]

Since \( F_{\{\alpha_{j-1}\}}(G_{j-1}) \neq 1 \), we see that for the constructed sequence of the groups
\[ G, G_1, G_2, G_3, \ldots, G_n, \ldots \]

it follows that
\[ |G| > |G_1| > |G_2| > |G_3| > \ldots > |G_n| > \ldots. \]

Since the group \( G \) is finite, we obtain \( G_k = 1 \) for some number \( k \). But
\[ G_k = G_{k-1}/F_{\{\alpha_{k-1}\}}(G_{k-1}). \]

This implies that \( F_{\{\alpha_{k-1}\}}(G_{k-1}) = G_{k-1}, \) a contradiction.

Thus, our assumption is not true and \( \mathcal{H} \subset \mathcal{L} \). Hence \( \mathcal{H} = \mathcal{L} \). The theorem is proved.

Note that theorem 2 gives an affirmative answer to question of A. Tzarev on distributivety of the lattice of all totally \( \sigma \)-local formations of finite groups [11, question 3.2].

Let \( \mathfrak{F} \) and \( \mathfrak{M} \) be totally \( \sigma \)-local formations such that \( \mathfrak{M} \subseteq \mathfrak{F} \), then \( \mathfrak{F}/\sigma \mathfrak{M} \) denotes the lattice of all totally \( \sigma \)-local formations between \( \mathfrak{M} \) and \( \mathfrak{F} \).

**Corollary 3.** Let \( \mathfrak{F} \) and \( \mathfrak{H} \) be totally \( \sigma \)-local formations. Then the lattice isomorphism holds
\[ \mathfrak{F} \lor \mathfrak{H} = \mathfrak{F}/\sigma \mathfrak{H} \land \mathfrak{H}. \]

In the case when \( \sigma = \sigma_1 \), we get from theorem 2 the following known results.

**Corollary 4** [16]. The lattice \( \ell_\omega \) of all totally local formations is distributive.

**Corollary 5** [2, p. 169]. The lattice of all soluble totally local formations is distributive.

**Corollary 6** [17]. The lattice \( \ell_\omega \) of all totally local formations is modular.

Recall that if \( \mathfrak{F} \) and \( \mathfrak{M} \) are totally local formations, \( \mathfrak{M} \subseteq \mathfrak{F} \), then by the symbol \( \mathfrak{F}/\omega \mathfrak{M} \) denotes the lattice of all totally local formations between \( \mathfrak{M} \) and \( \mathfrak{F} \).

**Corollary 7** [17]. Let \( \mathfrak{F} \) and \( \mathfrak{M} \) be totally local formations. Then we have
\[ \mathfrak{F} \lor_\omega \mathfrak{M} = \mathfrak{F}/_\omega \mathfrak{M} \cap \mathfrak{M}. \]