Дифференциальные уравнения и оптимальное управление

Differential equations and optimal control

ЗАДАЧА ОПТИМИЗАЦИИ ДЛЯ ОДНОГО КЛАССА ДИФФЕРЕНЦИАЛЬНО-РАЗНОСТНЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

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В работе исследуется линейная дифференциально-разностная система с запаздывающим аргументом. Такие системы имеют различные сферы применения, в том числе и повторяющиеся процессы с обучением. Для определения условий оптимальности управления в задаче приведения траектории системы в состояние покоя за минимальное время использовалась теорема об отделимости выпуклых множеств. Аналитические выражения для оптимального управления выведены для специального случая интегральных ограничений на управление. В целях демонстрации полученных результатов приведен иллюстративный пример с детальным вычислением основных элементов оптимального управления.

Ключевые слова: дифференциально-разностные системы; запаздывающий аргумент; задача оптимального быстродействия.

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OPTIMISATION PROBLEM FOR SOME CLASS OF HYBRID DIFFERENTIAL-DIFFERENCE SYSTEMS WITH DELAY

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In the paper, the linear differential-difference dynamic systems with delayed arguments are considered. Such systems have a lot of application areas, in particular, processes with repetitive and learning structure. We apply the method of the separation hyperplane theorem for convex sets to establish optimality conditions for the control function to drive the trajectory to zero equilibrium state in the fastest possible way. For the special case of the integral control constraints, the proposed method is detailed to establish an analytical form of the optimal control function. The illustrative example is given to demonstrate the obtained results with the step-by-step calculation of the basic elements of the optimal control.

Keywords: differential-difference system; delayed argument; time optimal control problem.

Introduction

The time delayed dynamic is frequently encountered in modern control system theory [1]. Differential-difference processes with delayed arguments (hybrid continuous-discrete) are a class of systems of both theoretic and applications interest [2]. Application areas include a lot of physical [3] and industrial processes [4], especially, with repetitive and learning structure [5] and others. Moreover, it is already known that [6] links between some types of linear repetitive processes and delay systems, which can, where appropriate, be used to great effect in the control related analysis of these processes. This paper is based on the work [7] and gives some new results on optimisation theory for delayed differential-difference linear processes. There are only a few research works in the literature devoted to optimisation theory (see, for example [8], and references therein). In this paper, we have adopted the method [9] based on the separation theorem for the convex sets to establish optimality conditions for the time optimal control problem. Then, we applied the classic approach from calculus of variations theory to study the structure of the optimal control for the sub-class of the delayed hybrid differential-algebraic processes. Furthermore, the proposed method is detailed for the special case of the integral control constraints where the applicable form of the optimal control function to drive the process dynamics to zero equilibrium state in the fastest possible way is established. It has been conjectured that such a setting is appropriate for development of the numerical methods for optimal control problems and related studies on for which very little work has been reported to date. The illustrative example, given in the paper, demonstrates the main stages and the step-by-step calculation to realise the analytical solution based on the obtained results to design the time optimal control function. This fact is interesting from theoretical and, as well, practical viewpoints. Some areas for short to medium term further research are also briefly discussed.

Notation. R^n denotes the n – dimensional Euclidean space, g^T and A^T mean transposed vector and matrix, respectively, I_n is the $n \times n$ identity matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

Optimisation problem for delayed differential-difference system

In the paper, we consider the linear system described by the pair of time delay differential and difference equations

$$\dot{x}(t) = Ax(t) + A_{-1}x(t-h) + B_0 y(t) + B_{-1}y(t-h) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + C_{-1}x(t-h) + D_{-1}y(t-h) + Du(t), \quad t \in [0, \alpha] \tag{2}$$

with initial conditions

$$x(t) = f(t), \ t \in [-h, 0), \ x(0) = x_0, \ y(t) = g(t), \ t \in [-h, 0] \tag{3}$$

and $x \in R^n$, $y \in R^m$, $u \in R^r$, where α, h are given real numbers such that $h < \alpha$, $f(t)$, $g(t)$ are given continuous functions, $A, A_{-1}, B, B_0, B_{-1}, C, C_{-1}, D, D_{-1}$ are given matrixes of the appropriate dimensions. The class $U(\cdot)$ of the admissible control vectors $u(t)$, $t \in [0, \alpha]$, is the set of the all piecewise continuous functions such that $u(t) \in \Omega$, $t \in [0, \alpha]$, where Ω is a compact convex set in R^r. The pair of the functions $(x(t), y(t))$ is termed the solution of the system $(1)-(3)$ for the given control vector $u(t)$, if they satisfy the differential equation (1)

almost everywhere on the interval $[0, \alpha]$, $t \neq kh$, $k = 0, 1, 2, \ldots$ and the difference equation (2) for all $t \in [0, \alpha]$. (Here the right-hand derivative is assumed at the moment $t = 0$.) Under the assumptions made here, it can be shown that the solution $x(t)$ is absolutely continuous and also that $y(t)$ is piecewise continuous on the interval $[0, \alpha]$.

Consider the following time optimal control problem for the process (1)–(3). For the given initial data $x(t) = f(t)$, $t \in [-h, 0)$, $x(0) = x_0$, $y(t) = g(t)$, $t \in [-h, 0]$, it is required to find the minimal time T, $T \in [0, \alpha]$, and the control function $u(t)$, $t \in [0, T]$, such that the corresponding solution of the system (1)–(3) satisfies the following condition:

$$x(t) \equiv 0,\, t \in [T - h, T]. \tag{4}$$

We assume, in addition, that the set of the admissible controls $U(\cdot)$ is non-empty. We say that the control function $u(\cdot) \in U(\cdot)$ is T-admissible control for the system (1)–(3) if the corresponding trajectory satisfies the condition (4). The solution of the system (1)–(3) can be constructed (see, for example, [6; 7] and references therein) using the step-by-step, or recurrent, procedure for each sub-interval of the form $[ih, (i+1)h)$, $i = 0, 1, \ldots, q_\alpha$, where $q_\alpha = \left[\dfrac{\alpha}{h}\right]$ denotes the integer part of the fraction $\dfrac{\alpha}{h}$. For this purpose, function $F(t, \tau)$ is introduced as a solution of the following differential equation

$$\frac{\partial F(t, \tau)}{\partial \tau} = -\sum_{j=1}^{q_t+1} F\big(t, \tau + (j-1)h\big) H_j,\ F(t, \tau) \equiv 0,\ \forall \tau > t,\ F(t, t - 0) = I_n. \tag{5}$$

It can be shown that the following formula for the solutions of the system (1)–(3) is true:

$$x(t) = s(t, f, g, x_0) + \int_0^t S(t, \tau) u(\tau) d\tau,\ t \in [0, \alpha], \tag{6}$$

where

$$s(t, f, g, x_0) = F(t, 0) x_0 + \int_{-h}^0 F\big(t, \tau + (q_t + 1)h\big)\big[P_{q_t+1} f(\tau) + Q_{q_t+1} g(\tau)\big] d\tau +$$

$$+ \sum_{j=1}^{q_t+1} \int_{-h}^0 F\big(t, \tau + (j-1)h\big) H_j f(\tau) d\tau,\ S(t, \tau) = \sum_{j=1}^{q_t+1} F\big(t, \tau + (j-1)h\big) V_j,$$

and

$$y(t) = CF(t, 0) x_0 + \int_{-h}^0 CF(t, \tau) H_1 f(\tau) d\tau + \int_{-h}^0 CF(t, \tau + h)\big[P_1 f(\tau) + Q_1 g(\tau)\big] d\tau +$$

$$+ \int_0^t CF(t, \tau) V_1 u(\tau) d\tau + C_{-1} f(t - h) + D_{-1} g(t - h) + D u(t),\ \forall t \in [0, h),$$

$$y(t) = CF(t, 0) x_0 + \sum_{j=0}^{q_t-1} M_{j+1} F\big(t - (j+1)h, 0\big) x_0 + \sum_{l=0}^{q_t-1} \sum_{j=0}^{q_t-l} \int_{-h}^0 M_l F(t - lh, \tau + jh) H_{j+1} f(\tau) d\tau +$$

$$+ \sum_{l=0}^{q_t} \int_{-h}^0 M_l F\big(t - lh, \tau + (q_t + 1 - l)h\big)\big[P_{q_t+1-l} f(\tau) + Q_{q_t+1-l} g(\tau)\big] d\tau +$$

$$+ \int_0^t R(t, \tau) u(\tau) d\tau + W_{q_t-1} f(t - q_t h) + \sum_{j=0}^{q_t} G_j u(t - jh) + K_{q_t-1} g(t - q_t h),\ \forall t \geq h,$$

where

$$H_j = \big(B_0 D_{-1}^{j-1} + B_{-1} D_{-1}^{j-2}\big)(C_{-1} + D_{-1} C),\ j = 3, \ldots, q_t + 1,\ H_1 = A + B_0 C,$$

$$H_2 = A_{-1} + B_0(C_{-1} + D_{-1} C) + B_{-1} C,\ V_j = \big(B_0 D_{-1}^{j-1} + B_{-1} D_{-1}^{j-2}\big) D,\ j = 2, \ldots, q_t + 1,$$

$$V_1 = B + B_0 D,\ Q_i = \big(B_0 D_{-1} + B_{-1}\big) D_{-1}^{i-1},\ P_i = \big(B_0 D_{-1} + B_{-1} D_{-1}^{i-2}\big) C_{-1},\ P_1 = A_{-1} + B_0 C_{-1},$$

$$R(t, \tau) = \sum_{l=0}^{q_t - 1} \sum_{j=0}^{q_t - l} M_l F(t - lh, \tau + jh) V_{j+1}, \; q_t = \left[\frac{t}{h} \right],$$

$$M_{j+1} = D_{-1}^j (C_{-1} + D_{-1} C), \; M_0 = C, \; G_j = D_{-1}^j D, \; K_i = D_{-1}^i C_{-1}, \; W_i = D_{-1}^{i+1}, \; j = 0, 1, \ldots, q_t.$$

Let $C^n[-h, 0]$, $h > 0$, denote the vector space of the continuous n vector function $f : [-h, 0] \to R^n$. Put

$$Z = \left\{ x \in R^n \big| x = s(T - h, f, g, x_0) \text{ for all } (f, g, x_0) \in C_{[-h, 0)} \times C_{[-h, 0]} \times \mathbb{R}^n \right\}$$

and

$$\mathcal{R} = \left\{ s \in Z \big| \text{ such that for } x = s \text{ there is } T\text{-admissible control } u(\cdot) \right\}. \tag{7}$$

In fact, the set $\mathcal{R}$ is the reachability set [7; 10] for the system $(1)-(3)$ with the additional condition. We assume that $\mathcal{R}$ is not empty, which is true if the system is controllable. In other words, we suppose that there exists at least one collection of the initial data

$$x(t) = f(t), \; t \in [-h, 0), \; x(0) = x_0, \; y(t) = g(t), \; t \in [-h, 0],$$

for which there exists the T-admissible control function. Additionally, we will assume next that $r = n$ and the matrices B, B_0, D such that there exist B^{-1} and $\left[E + DB^{-1} B_0 \right]^{-1}$ (this can be guaranteed, for example, by the appropriate spectrum assumptions for these matrices). Note that in this case the T-admissible control functions on the last interval $[T - h, T]$ is represented in the feedback form. Indeed, from (1) we have

$$u(t) = -B^{-1} \left[A_{-1} x(t - h) + B_0 y(t) + B_{-1} y(t - h) \right].$$

Substituting this into (2) yields

$$u(t) = Nx(t - h) + My(t - h), \; t \in [T - h, T], \tag{8}$$

where

$$N = \left[I + DB^{-1} B_0 \right]^{-1} \left[C_{-1} - DB^{-1} A_{-1} \right], \; M = \left[I + DB^{-1} B_0 \right]^{-1} \left[D_{-1} - DB_{-1}^2 \right]. \tag{9}$$

Thus, the considered problem is to determine the optimal control on the interval $[0, T - h]$.

We denote by $U_T(\cdot)$ the set of the all T-admissible control functions for the system $(1)-(3)$ corresponding to the set $\mathcal{R}$ in (7). By analogy with [9; 10] it can be shown that $\mathcal{R}$ is the convex set.

Theorem 1. *For the given initial data* $f(t)$, $t \in [-h, 0)$, $x(0) = x_0$, $g(t)$, $t \in [-h, 0]$, *the T-admissible control function for the system $(1)-(3)$ exists if, and only if, the following inequality is fulfilled:*

$$\max_{\|g\| = 1} \left\{ g^T s(T - h, f, g, x_0) + \inf_{u(\cdot) \in U_T(\cdot)} \int_0^{T-h} g^T S(T - h, \tau) u(\tau) d\tau \right\} \leq 0. \tag{10}$$

P r o o f. *Necessity.* Let T be the moment such that $u(t)$, $t \in [0, T]$, is a T-admissible control function for the system $(1)-(3)$. This means that the corresponding trajectory at the moment $t = T - h$ satisfies the condition $x(T - h) = 0$. Therefore,

$$s(T - h, f, g, x_0) + \int_0^{T-h} S(T - h, \tau) u(\tau) d\tau = 0. \tag{11}$$

Multiplying both sides of the last equality (11) by the vector $g \in R^n$ yields

$$g^T s(T - h, f, g, x_0) + \int_0^{T-h} g^T S(T - h, \tau) u(\tau) d\tau = 0.$$

Therefore,

$$g^T s(T - h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T - h, \tau) u(\tau) d\tau \leq 0. \tag{12}$$

Obviously, from (12) it follows (10).

Sufficiency. Let the inequality (10) hold for the given initial data (f, g, x_0). On contrary, assume that for this data there is no T-admissible control $u(\cdot)$, which solves the problem. This means that the corresponding vector $s^* = s^*(T - h, f, g, x_0)$ does not belong to the set $\mathcal{R}$, i. e. $s^*(T - h, f, g, x_0) \notin \mathcal{R}$. Since $\mathcal{R}$ is a convex set, there exists a supporting hyperplane with the non-trivial normal vector $g^* \in R^n$, $\|g^*\| = 1$ such that the following inequality

$$g^{*T} s^* > g^{*T} s, \ \forall s \in \mathcal{R}, \tag{13}$$

holds. Since $s \in \mathcal{R}$, there exists a T-admissible control function $u(t)$, $t \in [0, T - h]$, such that the corresponding trajectory at moment $t = T - h$ satisfies the condition $x(T - h) = 0$:

$$s + \int_0^{T-h} S(T - h, \tau) u(\tau) d\tau = 0.$$

Then $s = -\int_0^{T-h} S(T - h, \tau) u(\tau) d\tau$ and from (13) follows

$$g^{*T} s^* + \int_0^{T-h} g^{*T} S(T - h, \tau) u(\tau) d\tau > 0.$$

Since s is an arbitrary vector from the set $\mathcal{R}$, the last inequality is true for all $u(\cdot) \in U_T(\cdot)$. Therefore,

$$g^{*T} s^*(T - h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^{*T} S(T - h, \tau) u(\tau) d\tau > 0,$$

which contradicts (10). The theorem is proved.

Next, denote

$$\Lambda(T) = \max_{\|g\| = 1} \left\{ g^T s(T - h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T - h, \tau) u(\tau) d\tau \right\}. \tag{14}$$

It can be shown that $\Lambda(T)$ is a non-decreasing lower semicontinuous function, and hence we have the result bellow.

Theorem 2. *Given initial data $f(t)$, $t \in [-h, 0)$, $x(0) = x_0$, $g(t)$, $t \in [-h, 0]$, the moment T^0 is optimal if, and only if, T^0 is a minimal root of the equation*

$$\Lambda(T) = 0. \tag{15}$$

P r o o f. *Necessity.* Let T^0 and $u^0(\cdot)$ be the optimal solution for the optimisation problem. Then theorem 1 gives $\Lambda(T^0) \leq 0$. At the first, suppose that $\Lambda(T^0) < 0$. Since $\Lambda(T)$ is a non-decreasing and continuous function than $\exists \hat{T}$, $\hat{T} < T^0$, such that $\Lambda(T^0) \leq \Lambda(\hat{T}) \leq 0$. Then in accordance with theorem 1, the optimisation problem is solvable with $\hat{T} < T^0$ which is impossible. Thus, T^0 is the root of the equation (15). The minimality of T^0 can be shown analogously.

Sufficiency. Let the moment T^0 be the minimal root of $\Lambda(T) = 0$ for the control function $u^0(t)$, $t \in [0, T^0 - h]$. Suppose this control function is not optimal for the given initial data. Hence, there is the $\hat{T}$-admissible control function $\hat{u}(t)$, $t \in [0, \hat{T} - h]$, where $\hat{T} < T^0$. Then theorem 1 yields $\Lambda(\hat{T}) \leq 0$. However, noting non-decreasing function $\Lambda(T)$, we have $\Lambda(\hat{T}) \geq \Lambda(T^0) = 0$, which contradicts the minimality of the root T^0, which completes the proof.

Finally, the optimal time T^0 is given by the equality (15) and the optimal control function $u^0(t)$ is determined as

$$\min_{u \in U_T(\cdot)} \int_0^{T-h} g^{0T} S(T - h, \tau) u(\tau) d\tau = \int_0^{T-h} g^{0T} S(T - h, \tau) u^0(\tau) d\tau, \tag{16}$$

where g^0 is the vector which maximises the expression (14).

Time optimal control with integral constraints

The obtained general optimality conditions can be presented in a more applicable form for some special sets of admissible controls. Particularly in this section, the time optimal control problem for the system $(1)-(3)$ with the integral control constraints of the form

$$U(\cdot) \triangleq \left\{ u(\cdot) : \int_0^T u^T(\tau)u(\tau)d\tau \le 1 \right\} \tag{17}$$

is considered.

In accordance with theorem 2, formulas (6) and (16) the problem is to minimise the functional

$$\int_0^{T-h} g^T \sum_{j=1}^{q_{T-h}} F\big(T-h,\, \tau + (j-1)h\big)V_j u(\tau)d\tau \tag{18}$$

subject to the following integral constraint

$$\int_0^T u^T(\tau)u(\tau)d\tau \le 1. \tag{19}$$

Using (6) and (9) allows rewriting (19) as

$$\int_0^{T-h} u^T(\tau)u(\tau)d\tau + \int_{T-h}^{T} \big(Nx(\tau-h) + My(\tau-h)\big)^T \big(Nx(\tau-h) + My(\tau-h)\big)d\tau =$$

$$= Y + \int_0^{T-h} u^T(\tau)\big[I + \mathcal{G}(\tau)\big]^T u(\tau)d\tau + \int_0^{T-h} \big[\psi(\tau) + \varphi(\tau)\big]^T u(\tau)d\tau +$$

$$+ \int_0^{T-h} \int_0^{T-h} u^T(\tau)\big[\Psi(\tau,\theta) + \Phi(\tau,\theta)\big]u(\theta)d\tau \le 1, \tag{20}$$

where

$$Y = \int_{T-h}^{T} \big[Nx(\tau-h) + My(\tau-h)\big]^T \big[Nx(\tau-h) + My(\tau-h)\big]dt,$$

$$\psi(\tau) = \int_{T-2h}^{T-h} \left\{ S^T(\theta,\tau)N^T\big[Ns(\theta) + Mr(\theta)\big] + \big[s^T(\theta)N^T + r^T(\theta)M^T\big]MR(\theta,\tau) \right\}d\theta,$$

$$\varphi(\tau) = \begin{cases} \big(s^T(\tau)N^T + r^T(\tau)M^T\big)MG_0, \ \tau \in (T-2h,\, T-h], \\ \big(s^T(\tau+h)N^T + r^T(\tau+h)M^T\big)MG_1, \ \tau \in (T-3h,\, T-2h], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \big(s^T\big(\tau+(q_T-1)h\big)N^T + r^T\big(\tau+(q_T-1)h\big)M^T\big)MG_{q_T-1}, \ \tau \in [0,\, h], \end{cases}$$

$$\Phi(\theta,\tau) = \int_{T-2h}^{T-h} \left\{ S^T(t,\tau)N^T\big[NS(t,\theta) + MR(t,\theta)\big] + R^T(t,\tau)M^T MR(t,\theta) \right\}d\tau,$$

$$\mathcal{G}(\tau) = \begin{cases} G_0^T M^T M G_0 + G_0 M^T M G_1 e^{-ph} + \ldots + G_0^T M^T M G_{g_T-1}e^{-(q_T-1)ph}, \ \tau \in (T-2h,\, T-h], \\ G_1^T M^T M G_1 + G_1 M^T M G_2 e^{-ph} + \ldots + G_1^T M^T M G_{g_T-2}e^{-(q_T-2)ph}, \ \tau \in (T-3h,\, T-2h], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ G_{g_T-2}^T M^T M G_{g_T-2} + G_{g_T-2}^T M^T M G_{g_T-1}e^{-ph}, \ \tau \in (h,\, 2h], \\ G_{g_T-1}^T M^T M G_{g_T-1}, \ \tau \in [0,\, h], \end{cases}$$

$$\Psi(\tau) = \begin{cases} \left[S^T(\theta, \tau) N^T + R^T(\theta, \tau) M^T \right] MG_0, \ \tau \in (T - 2h, T - h], \\ \left[S^T(\theta + h, \tau) N^T + R^T(\theta + h, \tau) M^T \right] MG_1, \ \tau \in (T - 3h, T - 2h], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left[S^T(\theta + (q_T - 1)h, \tau) N^T + R^T(\theta + (q_T - 1)h, \tau) M^T \right] MG_{q_T - 1}, \ \tau \in [0, h]. \end{cases}$$

Here e^{-kph} denotes the shift operator such that $\left(e^{-kph}u \right)(\tau) = u(\tau - kh)$. Then, using the Lagrange multiplier method for (18), (19) leads to the functional

$$\Pi(u) = \int_0^{T-h} g^T S(T - h, \tau) u(\tau) d\tau + \lambda \left\{ Y + \int_0^{T-h} u^T(\tau) \left[I_m + \mathcal{G}(\tau) \right]^T u(\tau) d\tau + \right.$$
$$\left. + \int_0^{T-h} \left[\psi(\tau) + \varphi(\tau) \right]^T u(\tau) d\tau + \int_0^{T-h} \int_0^{T-h} u^T(\tau) \left[\Psi(\tau, \theta) + \Phi(\tau, \theta) \right] u(\theta) d\tau \right\}, \tag{21}$$

which is subject of minimisation with respect to unknowns λ and $u(t)$. Then the first variation $\delta\Pi$ for $\Pi(u)$, given by (21) can be represented as

$$\delta\Pi(u) = \left. \frac{\delta\Pi(u + \alpha v)}{\partial \alpha} \right|_{\alpha = 0} = \int_0^{T-h} v(\tau) S(Th, \tau) g d\tau +$$
$$+ \int_0^{T-h} \lambda \left\{ u^T(\tau) \left[I + \mathcal{G}(\tau) \right]^T d\tau + \left[\psi(\tau) + \varphi(\tau) \right]^T + \int_0^{T-h} u(\tau) K(\theta, \tau) u(\theta) d\theta \right\}^T v(\tau) d\tau, \tag{22}$$

where

$$K(\theta, \tau) = \left(\Psi(\tau, \theta) + \Phi(\tau, \theta) \right) + \left(\Psi^T(\tau, \theta) + \Phi^T(\tau, \theta) \right).$$

The Lagrange multiplier method yields that the optimal solutions satisfy to the equality $\delta\Pi(u) = 0$ for all functions $v(\tau)$. Hence, from (22) it follows

$$S^T(T - h, \tau) g + \lambda \left\{ 2 \left[I_m + \mathcal{G}(\tau) \right] u(\tau) + \psi(\tau) + \varphi(\tau) + \int_0^{T-h} K(\theta, \tau) u(\theta) d\theta \right\} = 0. \tag{23}$$

The solution of (23) can be represented as

$$u_g(t) = u_1(t) + u_2(t), \text{ where } u_2(t) = \frac{1}{\lambda} L(t) g. \tag{24}$$

Here the vector $u_1(t)$ and the $(n \times r)$-matrix $L(t)$ satisfy the following integral equations

$$\int_0^{T-h} K(\theta, t) u_1(\theta) d\theta + 2u_1(t) \left(I + \mathcal{G}(t) \right) + \psi(t) + \varphi(t) = 0, \ 2L(t) + S(T - h, t) + \int_0^{T-h} K(\theta, t) L(\theta) d\theta = 0.$$

To show this it is sufficient to substitute (24) into (23), which gives

$$S^T(T - h, \tau) g + \lambda \left\{ 2 \left(u_1(t) + \frac{L(t)g}{\lambda} \right) \left[I_m + \mathcal{G}(\tau) \right] + \int_0^{T-h} u(\tau) K(\theta, \tau) \left(u_1(\theta) + \frac{L(\theta)g}{\lambda} \right) d\theta \right\} +$$
$$+ \psi(\tau) + \varphi(\tau) = \left(S^T(T - h, \tau) + L(\tau) + \int_0^{T-h} K(\theta, \tau) L(\theta) d\theta \right) g +$$
$$+ \lambda \left[2u_1(\tau) \left(I_m + \mathcal{G}(\tau) \right) + \psi(\tau) + \varphi(\tau) + \int_0^{T-h} u^T(\tau) K(\theta, \tau) u_1(\theta) \right] = 0.$$

The unknown multiplier λ can be determined by the fact that the required control function satisfies the condition $\int_0^T u^T(\tau)u(\tau)d\tau = 1$. Hence, using (20) yields

$$Y + \int_0^{T-h}\left[\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g\right]^T\left(I_m + \mathcal{G}(\tau)\right)\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g\right]\right]d\tau +$$

$$+ \int_0^{T-h}\left(\psi(\tau) + \varphi(\tau)\right)^T\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g\right]d\tau +$$

$$+ \int_0^{T-h}\int_0^{T-h}\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g\right]^T\left(\Psi(\theta,\tau) + \Phi(\theta,\tau)\right)\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g\right]d\theta d\tau = 1.$$

This leads to the following equation for λ: $a\dfrac{1}{\lambda^2} + 2b\dfrac{1}{\lambda} + c = 0$, where the required coefficients are

$$a = \int_0^{T-h} g^T L(\tau)^T L(\tau)g\,d\tau + \int_0^{T-h}\int_0^{T-h} g^T L^T(\tau)K(\theta,\tau)L(\theta)d\theta d\tau, \tag{25}$$

$$b = \int_0^{T-h} u_1^T(\tau)L(\tau)g\,d\tau + \int_0^{T-h}\left[\psi(\tau) + \varphi(\tau)\right]L(\tau)g\,d\tau + \int_0^{T-h}\int_0^{T-h} g^T L^T(\tau)K(\theta,\tau)u_1(\theta)d\tau d\theta,$$

$$c = Y - 1 + \int_0^{T-h} u_1^T(\tau)\left(I + \mathcal{G}(\tau)\right)u_1(\tau)d\tau + 2\int_0^{T-h}\left(\psi(\tau) + \varphi(\tau)\right)u_1(\tau)d\tau +$$

$$+ \int_0^{T-h}\int_0^{T-h} u_1^T(\tau)\left[\Phi(\theta,\tau) + \Psi(\theta,\tau)\right]u_1(\theta)d\tau d\theta. \tag{26}$$

Thus, the required λ is the positive root of this equation, and the optimal control for the given T is defined by the formula (24). Substituting the obtained control function $u_g(t)$ of (24) into (14) and noting theorem 2 we have shown that the optimal control function for the considered integral constraints (17) and the given above assumptions can be presented by the following theorem.

Theorem 3. *Optimal time T^0 for the optimisation problem (1)–(3) with integral constraints (17) is the minimal root of the equality*

$$\max_{\|g\|=1}\left\{g^T s(T-h, f, g, x_0) + \int_0^{T-h} g^T S(T-h, \tau)u_g(\tau)d\tau\right\} = 0 \tag{27}$$

and the corresponding optimal control is given as

$$u^0(t) = \begin{cases} u_{g^0}(t), & t \in \left[0, T^0 - h\right), \\ Nx^0(t-h) + My^0(t-h), & t \in \left[T^0 - h, T^0\right], \end{cases} \tag{28}$$

where $u_g(t)$ is given by (24), the vector g^0 realises maximum in (27) and the matrices M, N are defined by formulas (8) and (9).

It should be noted that the proposed method can be applied for solution of constrained optimisation problems with different types of the cost functional. The following example demonstrates the possibility of such application.

Example. The given illustrative example shows the practical steps to realise the analytical calculations based on the obtained results to solve the optimisation problem with an energy cost functional. In order to demonstrate the main stages of these calculations we consider the following equation with control input

$$\dot{x}(t) = -x\left(t - \frac{\pi}{2}\right) + u(t),\ t \in \left[0, \frac{3\pi}{2}\right], \tag{29}$$

and the initial data

$$x(t) \equiv 0, \ t \in \left[-\frac{\pi}{2}, 0\right), \ x(0) = 1. \tag{30}$$

The considered example is a particular case of the system $(1)-(3)$, where

$$A_{-1} = 1, \ B = 1, \ h = \frac{\pi}{2}, \ T = \frac{3\pi}{2}, \ U = R^1, \ r = 1$$

and other coefficients are trivial.

Consider the following problem: minimise the cost functional

$$J(u) \to \min_u, \ J(u) = \int_0^{3\pi/2} u^2(t) dt \tag{31}$$

over the solutions of (29), (30) subject to the constraints of the form

$$\int_0^T u^T(\tau) u(\tau) d\tau \leq 1, \ x(t) \equiv 0, \ t \in \left[\pi, \frac{3\pi}{2}\right]. \tag{32}$$

Let $M^0 = J(u^0)$ be the optimal cost value for the problem (29), (30). Consider the following time optimal problem: minimise

$$T \to \min \tag{33}$$

over the solutions of the control system

$$\dot{x}(t) = -x\left(t - \frac{\pi}{2}\right) + u(t), \ t \in [0, T], \tag{34}$$

with the initial conditions (30) and the constraints of the form

$$\int_0^T u^2(t) dt \leq M^0, \ x(t) \equiv 0, \ t \in \left[T - \frac{\pi}{2}, T\right]. \tag{35}$$

It is easy to show that the optimal solution for the problem $(33)-(35)$ is $T^0 = \frac{3}{2}\pi$. Thus, the optimisation problems $(29)-(32)$ and $(33)-(35)$ are equivalent. Hence, the proposed method can be used for solution of the given problem $(29)-(32)$.

Step 1: representation of equation solution.

In our case, the function $F(t, \tau)$ from the formula (5) satisfies the equation

$$\frac{\partial F(t, \tau)}{\partial \tau} = F\left(t, \tau + \frac{\pi}{2}\right), \ F(t, \tau) \equiv 0, \ \tau > t, \ F(t, t) = 1. \tag{36}$$

It is easy to check that the function

$$F(t, \tau) = \begin{cases} e^{-i(t-\tau)}, & \text{if } \tau \leq t, \\ 0, & \text{if } \tau > 0, \end{cases} \tag{37}$$

is the solution of the equation (36), where i in (37) means imaginary unit ($i^2 = -1$). Thus, the solution of the system (29) with the initial data (30) for $t \in \left[0, \frac{3\pi}{2}\right]$ is given as

$$x(t) = F(t, 0) x(0) + \int_0^t F(t, \tau) u(\tau) d\tau = e^{-it} + \int_0^t e^{-i(t-\tau)} u(\tau) d\tau = e^{-it} \left(1 + \int_0^t e^{i\tau} u(\tau) d\tau\right). \tag{38}$$

The problem statement says that we exploit the real valued functions. Using the known formula for $e^{i\varphi}$ the corresponding real part of $x(t)$ in (38) can be extracted when it is necessary.

Step 2: structure of the optimal control.

In order to determine the optimal control function, we need to calculate the functions $\Phi(\tau, \theta)$, $K(\tau, \theta)$ and other constants γ, a, b, c which are required in (28). Note that in our case $H_1 = 0$, $H_2 = 1$, $H_j = 0$, $V_1 = 1$, $V_j = 0$, $M_j = 0$, $\varphi(\tau) = 0$, $\psi(t) = \int_{\pi/2}^{\pi} e^{-i(t-\tau)} x(0) e^{-it} dt = -i e^{i\tau}$. Since the problem is given in real valued terms, we are needed to pick the real parts in the obtained functions $\operatorname{Re} \psi(\tau) = \sin \tau$, $\operatorname{Im} \psi(\tau) = -\cos \tau$. Further,

$$K(\tau, \theta) = 2 \int_{\pi/2}^{\pi} F(t, \tau) F(t, \theta) dt = \begin{cases} i \cos \theta \sin \tau - \sin \theta \sin \tau, \ \tau \geq \theta, \ \dfrac{\pi}{2} \leq \tau, \ \theta \leq \dfrac{3\pi}{2}, \\ i \cos \tau \sin \theta - \sin \tau \sin \theta, \ \tau > \theta, \ \dfrac{\pi}{2} \leq \tau, \ \theta \leq \dfrac{3\pi}{2}, \\ 0, \ 0 \leq \tau, \ \theta \leq \dfrac{\pi}{2}. \end{cases}$$

Thus, the real part of the function K is $K(\tau, \theta) = -\sin \theta \sin \tau$ for $\dfrac{\pi}{2} \leq \tau, \theta \leq \dfrac{3\pi}{2}$. It is easy to see that $K(\tau, \theta) = K(\theta, \tau)$. According to the formula (28), the optimal control function for $t \in \left[0, \dfrac{3\pi}{2}\right]$ is represented as follows

$$u_g(t) = v(t) + w(t), \text{ where } w(t) = \frac{1}{\lambda} L(t) g. \tag{39}$$

Step 3: solution of integral equations.

The scalar function $v(t)$, $t \in \left[0, \dfrac{3\pi}{2}\right]$, from (39) satisfies the following integral equation

$$2v(\tau) + 2\psi(\tau) + \int_0^{\pi} K(\tau, \theta) v(\theta) d\theta = 0. \tag{40}$$

If $\tau \in \left[0, \dfrac{\pi}{2}\right]$ then $K(\tau, \theta) \equiv 0$ and hence from (40) it follows that $v(\tau) = -\psi(\tau) = -\sin \tau$, $\tau \in \left[0, \dfrac{\pi}{2}\right]$. If $\tau \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ then for the unknown function $v(\tau)$ we have the following integral equation

$$\sin \tau \int_0^{\pi} \sin \theta v(\theta) d\theta - v(\tau) = \sin \tau. \tag{41}$$

Denote $\int_0^{\pi} \sin \theta v(\theta) d\theta \doteq W$. Multiplying (41) by $\sin \tau$ and integrating the obtained relation with respect to τ over the interval $[0, \pi]$, we have

$$\int_0^{\pi} \sin^2 \tau d\tau \int_0^{\pi} \sin \theta v(\theta) d\theta - \int_0^{\pi} \sin \tau v(\tau) d\tau = \int_0^{\pi} \sin^2 \tau d\tau.$$

Noting that $\int_0^{\pi} \sin^2 \tau d\tau = \dfrac{\pi}{2}$ and $\int_0^{\pi} \sin \tau v(\tau) d\tau = W$ we have the following algebraic equation with respect to the unknown value W: $W\left(\dfrac{\pi}{2} - 1\right) = \dfrac{\pi}{2}$ and hence $W = \dfrac{\pi}{\pi - 2}$. Then from (41) we have $v(\tau) = W \sin \tau - \sin \tau =$

$$= \frac{2}{\pi - 2} \sin \tau \text{ and } v(\tau) = \begin{cases} -\sin \tau, \ \tau \in \left[0, \dfrac{\pi}{2}\right), \\ \dfrac{2}{\pi - 2} \sin \tau, \ \tau \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]. \end{cases}$$

The function $L(\tau)$ of (39) satisfies to the following integral equation:

$$2L(\tau) + \int_0^{\pi} K(t, \theta) L(\theta) d\theta + S(\pi, \tau) = 0.$$

In our case $S(\pi, \tau) = \begin{cases} -\cos \tau, \ \tau \leq \pi, \\ 0, \ \tau > \pi. \end{cases}$

Analogously to $v(\tau)$ it can be shown that the required function is $L(\tau) = \begin{cases} \cos \tau, \ \tau \in [0, \pi], \\ 0, \ \tau \in \left[\pi, \dfrac{3\pi}{2}\right]. \end{cases}$

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Hence, the optimal control function is given as $u_g(t) = v(t) + \dfrac{1}{\lambda} L(t) g$. Here, the parameter λ is determined as the positive solution of the following algebraic equation $a \dfrac{1}{\lambda^2} + 2b \dfrac{1}{\lambda} + c = 0$, where

$$-a = \frac{1}{2} \int\limits_{0}^{T-h} S(T-h, \tau) L(\tau) d\tau = \frac{1}{2} \int\limits_{0}^{\pi} \cos^2 \tau \, d\tau = \frac{1}{4}\pi, \quad c = \gamma - 1 + \int\limits_{0}^{T-h} (\psi(\tau) + \varphi(\tau)) v(\tau) d\tau =$$

$$= \gamma - 1 + \int\limits_{0}^{\pi/2} \sin\tau \, 2\sin\tau \, d\tau + \int\limits_{\pi/2}^{\pi} \sin\tau \frac{2}{\pi - 2} \sin\tau \, d\tau = \gamma + \frac{\pi - 1}{2}, \quad b = 0.$$

Here, $\gamma = \int\limits_{T-h}^{T} \big[Ns(\tau-h, x_0) + Mr(\tau-h) \big]^T \big[Ns(\tau-h, x_0) + Mr(\tau-h) \big] dt$, where the coefficients M, N are given by (9). Since $r(\tau) = 0$, $s(t, x_0) = \int\limits_{\pi/2}^{\pi} e^{-it} e^{it} dt = i$, $\gamma = i$ then the required in (25), (26) parameters are $c = \dfrac{\pi - 1}{2}$, $b = 0$, $a = -\dfrac{\pi}{4}$. Hence, the required positive root is $\lambda^* = \sqrt{\dfrac{\pi}{2\pi - 2}}$. Thus, the optimal control function is given as

$$u_g(t) = \begin{cases} \sin t + \dfrac{g}{\lambda^*} \cos t, & t \in \left[0, \dfrac{\pi}{2}\right], \\[2.5ex] \dfrac{\pi}{\pi - 2} \sin t + \dfrac{g}{\lambda^*} \cos t, & t \in \left[\dfrac{\pi}{2}, \pi\right], \\[2.5ex] \dfrac{\pi}{\pi - 2} \sin t + 0, & t \in \left[\pi, \dfrac{3\pi}{2}\right]. \end{cases}$$

Step 4: minimal root of the equality (27).

According to (28), the unknown g is determined as the maximising element in the equality $\max\limits_{\|g\|=1} \mathcal{L}(g, T) = 0$, where $\mathcal{L}(g, T) = gs(T-h, x_0) + \int\limits_{0}^{T-h} gF(T-h, \tau) u_g(\tau) d\tau$. In our case, we have $\max\limits_{g \neq 0} \Big\{ g \cos \pi +$

$+ \int\limits_{0}^{\pi} gF(\pi, \tau) u_g(\tau) d\tau \Big\} = 0$. After simplifying, we have

$$\max\limits_{g \neq 0} \left\{ -g + \int\limits_{0}^{\pi/2} g(-\cos\tau) \left(\sin\tau + \frac{g}{\lambda^*} \cos\tau \right) d\tau + \int\limits_{\pi/2}^{\pi} g(-\cos\tau) \left(\frac{\pi}{\pi - 2} \sin\tau + \frac{g}{\lambda^*} \cos\tau \right) d\tau \right\} =$$

$$= \max\limits_{g \neq 0} \left\{ g \left(-\frac{3}{2} + \frac{\pi}{2(\pi - 2)} \right) - g^2 \left(\frac{\pi}{2\lambda^*} \right) \right\} = \max\limits_{\|g\|=1} \left\{ -g^2 \sqrt{1 + \frac{2}{\pi}} - g \frac{2\pi - 6}{2\pi - 4} \right\} = 0.$$

The maximising element for this equality is $g^0 = \dfrac{6 - 2\pi}{2\pi - 4} \sqrt{\dfrac{\pi}{\pi + 2}}$. Finally, the optimal control is given as

$$u_g^0(t) = \begin{cases} \sin t + \dfrac{6 - 2\pi}{2\pi - 4} \sqrt{\dfrac{2\pi - 2}{\pi + 2}} \cos t, & t \in \left[0, \dfrac{\pi}{2}\right], \\[2.5ex] \dfrac{\pi}{\pi - 2} \sin t + \dfrac{6 - 2\pi}{2\pi - 4} \sqrt{\dfrac{2\pi - 2}{\pi + 2}} \cos t, & t \in \left[\dfrac{\pi}{2}, \pi\right], \\[2.5ex] \dfrac{\pi}{\pi - 2} \sin t, & t \in \left[\pi, \dfrac{3\pi}{2}\right]. \end{cases}$$

Conclusion

This work covers only the first attempts to investigate the optimisation problems for the considered hybrid processes, and hence a rich material to be the subject for further work. In particular, our interest is the optimal control problem where the state variables at the final interval $[\alpha, \alpha + h]$ are equal to the pre-assigned functions $x(t) = \varphi(t)$, $y(t) = \psi(t)$, $t \in [\alpha, \alpha + h]$ (see, also, [11]). Also, it is worth mentioning here that the approach of the supporting control functions setting described in [12] can be used for the design of the numerical algorithms with good conditioning properties. These problems are subject of ongoing work and will be reported in due course.

References

1. Wang J, He M, Xi J, Yang X. Suboptimal output consensus for time-delayed singular multi-agent systems. *Asian Journal of Control*. 2018;20(11):721–734. DOI: 10.1002/asjc.1592.

2. Hale JK, Lunel SMV. *Introduction to functional differential equations*. New York: Springer-Verlag; 1993. 450 p. (Bloch A, Epstein CL, Goriely A, Greengard L, editors. Applied mathematical sciences; volume 99). DOI: 10.1007/978-1-4612-4342-7.

3. Grigorieva EV, Kaschenko SA. *Asymptotic representation of relaxation oscillations in lasers*. Switzerland: Springer International Publishing; 2017. 230 p. DOI: 10.1007/978-3-319-42860-4.

4. Zhang C, Wang X, Wang C, Zhou W. Synchronization of uncertain complex networks with time-varying node delay and multiple time-varying coupling delays. *Asian Journal of Control*. 2018;20(1):186–195. DOI: 10.1002/asjc.1539.

5. Rogers E, Galkowski K, Owens DH. *Control systems theory and applications for linear repetitive processes*. Berlin: Springer Verlag; 2007. 456 p. (Allgöwer F, Morari M, editors. Lecture notes in control and information sciences; volume 349). DOI: 10.1007/978-3-540-71537-5.

6. Dymkou S, Rogers E, Dymkov M, Galkowski K, Owens DH. Delay systems approach to linear differential repetitive processes. *IFAC Proceedings Volumes*. 2003;36(19):333–338. DOI: 10.1016/S1474-6670(17)33348-7.

7. Dymkou S, Rogers E, Dymkov M, Galkowski K, Owens DH. An approach to controllability and optimization problems for repetitive processes. In: *Stability and control processes (SCP-2005). Proceedings of international conference; Saint Petersburg, Russia. Volume II*. [S. l.]: Saint Petersburg University; 2005. p. 1504–1516.

8. Dymkou S. Graph and 2-D optimization theory and their application for discrete simulation of gas transportation networks and industrial processes with repetitive operations [dissertation]. Aachen: RWTH; 2006. 147 p.

9. Marchenko VM. Hybrid discrete-continuous systems. II. Controllability and reachability. *Differential Equations*. 2013;49(1):112–125.

10. Gabasov R, Kirillova FM. *The qualitative theory of optimal processes*. New York: M. Dekker; 1976. 640 p.

11. Dymkou S, Dymkov M, Rogers E, Galkowski K. Optimal control of non-stationary differential linear repetitive processes. *Integral Equations and Operator Theory*. 2008;60:201–216.

12. Dymkov M, Rogers E, Dymkou S, Galkowski K. Constrained optimal control theory for differential linear repetitive processes. *SIAM Journal on Control and Optimization*. 2008;47(1):396–420. DOI: 10.1137/060668298.