ON SOLUTIONS OF THE CHAZY EQUATION

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The Chazy system determines the necessary and sufficient conditions for the absence of movable critical points of solutions of the particular third order differential equation that was considered by Chazy in one of the first papers on the classification of higher-order ordinary differential equations with respect to the Painlevé property. The solution of the complete Chazy system in the case of constant poles has been already obtained. However, the question of integrating the Chazy equation remained open until now. In this paper, we prove that in the case of constant poles, under some additional conditions, this equation is integrated in elliptic functions.

Keywords: Chazy equation; Chazy system; Painlevé property; elliptic functions.
Introduction

It is known that the Painlevé equations appeared as a result of solving the classification problem regarding the Painlevé property for the second order ordinary differential equations [1; 2]. For the equations of higher orders, the Painlevé problem, which consists in determining the conditions for the absence of movable multi-valued singularities for solutions, in the general case remains open. At present, the most complete results have been obtained only for polynomial equations of higher orders. So, for example, equations of the form

\[ y''' = P(z, y, y', y''), \]

where \( P(z, y, y', y'') \) is a polynomial of \( y \) and its derivatives are considered in [3–10]. Some classes of equations of the fourth and higher orders were studied in [11; 12].

One of the first works on the classification of higher-order equations with respect to the Painlevé property was a paper by Chazy [3]. It deals with the Painlevé property of the equation

\[ y'' + \sum_{k=1}^{6} \frac{(y' - a'_k)^3}{y - a_k} + \sum_{k=1}^{6} \frac{A_k(y' - a'_k)^2 + B_k(y' - a'_k)}{y - a_k} + \sum_{k=1}^{6} \frac{C_k(y' - a'_k)}{y - a_k} + Dy''' + Ey' + \sum_{k=1}^{6} \frac{F_k(y - a_k)}{y - a_k}, \]

(1)

where the poles \( a_k = a_k(z) \) are finite, distinct and in general are functions of the independent variable \( z \).

The paper [3] also contains the system of 31 algebraic and differential equations

\[ (A) \quad \sum_{j=1}^{6} A_j = 0, \quad \sum_{j=1}^{6} a_j A_j = -6, \quad \sum_{j=1}^{6} a_j^2 A_j = -2 \sum_{j=1}^{6} a_j, \quad 2A_k^2 + \sum_{j=1}^{6} \frac{A_k - A_j}{a_k - a_j} = 0, \quad k = 1, \ldots, 6 \ (j \neq k), \]

\[ (B) \quad \sum_{j=1}^{6} \left( B_k - B_j \right) \left( -\frac{A_k}{2} - \frac{1}{a_k - a_j} \right) + A_k^2 - \sum_{j=1}^{6} \frac{a_k^2}{a_k - a_j} - \left( a_k^2 - a_j^2 \right) + \sum_{j=1}^{6} \left( \frac{3A_j}{a_k - a_j} \right)^2 + \left( \frac{B_k - B_j}{a_k - a_j} \right)^2 + \frac{A_k^2 - A_j^2}{a_k - a_j} - \frac{B_k^2 + B_j^2 - B_k B_j D + E}{a_k - a_j} = 0, \]

\[ (C) \quad 2D + \sum_{j=1}^{6} \left( B_j - 3a_j A_j \right) = 0, \]

\[ (D) \quad \sum_{j=1}^{6} F_j = 0, \quad \sum_{j=1}^{6} a_j F_j = 0, \quad \sum_{j=1}^{6} a_j^2 F_j = 0, \quad -a_k^2 C_k + C_k^2 + D(a_k^2 - C_k) + E a_k^2 + F_k \prod_{j=1}^{6} (a_k - a_j) + \sum_{j=1}^{6} A_k \left( a_k^2 - a_j^2 \right) + B_k \left( a_k^2 - a_j^2 \right)^2 + \left( C_k - C_j \right) \left( a_k^2 - a_j^2 \right) + (a_k^2 - a_j^2) \left( a_k'' - a_j'' \right) = 0, \]

for 26 unknown functions \( A_k = A_k(z), B_k = B_k(z), C_k = C_k(z), D = D(z), E = E(z), F_k = F_k(z) \). In [3, p. 367–369] Chazy claimed that the solution of this system determines necessary and sufficient conditions for the absence of movable critical points of solutions of the equation (1). In this case, the poles \( a_k \) are the parameters of the system. The equations of the system \((A) - (F)\) we denote below by \( (A_1), \ldots, (A_6), (B_1), \ldots, (B_6), (C_1), \ldots, (C_6), (D), (F_1), \ldots, (F_9) \).

The solution of the Chazy \( A \)-system was obtained in [13] and the solution of the complete Chazy system in the case of constant poles \( a_k \) in expanded form is given in [14]. However, the question of integrating the Chazy equation (1) remained open until now. In this paper, we prove that in the case of constant poles, under some additional conditions, the equation (1) is integrated in elliptic functions.

Let us briefly summarise some results from [13; 14] which we need to obtain the main result stated in theorem 3.
Solution of system \((A)\)

First, let us note that the successive elimination of the variables \(A_4\) allows us uniquely express \(A_6, A_5, A_4, A_3, A_2\) in terms of \(A_1\). Indeed from the system \((A)\) we successively obtain

\[
\begin{align*}
A_6 &= -A_5 - A_4 - A_3 - A_2 - A_1, \\
A_5 &= -\frac{6}{a_{56}} A_6 - \frac{a_{46}}{a_{56}} A_4 - \frac{a_{36}}{a_{56}} A_3 - \frac{a_{26}}{a_{56}} A_2 - \frac{a_{16}}{a_{56}} A_1, \\
A_4 &= 2\left(\frac{a_{41} + a_{42} + a_{43} - a_{45} - a_{46}}{a_{45}a_{46}}\right) - \frac{a_{43}a_{46}}{a_{45}a_{46}} A_3 - \frac{a_{23}a_{46}}{a_{45}a_{46}} A_2 - \frac{a_{35}a_{46}}{a_{45}a_{46}} A_1, \\
A_3 &= 2a_{13}\left(-a_{12} - a_{13} + 2a_{14} + 2a_{15} + 2a_{16}\right) - \frac{a_{13}a_{14}a_{26}}{a_{13}a_{14}a_{36}} A_2 + \\
&+ \frac{a_{13}a_{14}a_{15}a_{16}}{a_{13}a_{14}a_{15}a_{36}} \left(\frac{1}{a_{12}} + \frac{1}{a_{13}} + \frac{2}{a_{14}} + \frac{2}{a_{15}} + \frac{2}{a_{16}}\right) A_1^2,
\end{align*}
\]

(2)

where \(a_{ij}\) denote the non-zero pole differences \(a_i - a_j\). Let’s also note the structure of \(A_2\):

\[
A_2 = \frac{n_0 A_4^2 + n_1 A_4 + n_2 A_2 + n_3 A_1 + n_4}{(a_2 - a_4)(a_2 - a_0)Q_2},
\]

(3)

where \(Q_2 = d_0 A_4^2 + d_1 A_4 + d_2\), and the coefficients \(n_0, n_1, n_2, n_3, n_4\) are determined through \(a_1, \ldots, a_6\).

After the substitution of the expressions thus obtained into the system \((A)\), the first four equations of this system become identities, and the equations \(\{A_5, A_6\}\) acquire the form

\[
\begin{align*}
\frac{a_{56}^2 u_{45} u_{46} u_{56} U}{a_{45} a_{46} a_{56} Q_2^2} &= 0, \\
\frac{a_{45} u_{46} u_{56} u_{26} U}{a_{45} a_{46} a_{56} Q_2^2} &= 0, \\
\frac{a_{45} u_{56} U}{a_{45} a_{46} a_{56} Q_2^2} &= 0, \\
\frac{a_{45} u_{56} U}{a_{45} a_{46} a_{56} Q_2^2} &= 0,
\end{align*}
\]

respectively, where \(a_{ij} = a_i - a_j\), \(u_i = 2a_i - 2a_i + (a_i - a_j) \left( a_i - a_j \right) A_i\); \(U\) and \(U_4, U_5, U_6\) are polynomials of the fifth and second degrees in \(A_1\), respectively.

Next two theorems from [14] follow from the successive considering of the two cases \((u_i = 0 \text{ and } U = 0)\).

**Lemma 1.** The system \((A)\) admits the symmetry \((A_k, a_k) \leftrightarrow (A_j, a_j), j, k = 1, \ldots, 6\).

This lemma shows that the permutation of arbitrary components \((A_k, a_k)\) of the solution of the system \((A)\) with arbitrary components \((A_j, a_j)\) leads to the solution of the system \((A)\).

**Theorem 1.** The system \((A)\) has the solution

\[
\begin{align*}
A_j &= \frac{1}{a_5 - a_j} + \frac{1}{a_6 - a_j}, \quad j = 1, \ldots, 4, \\
A_5 &= \frac{1}{a_1 - a_5} + \frac{1}{a_2 - a_5} + \frac{1}{a_3 - a_5} + \frac{1}{a_4 - a_5} + \frac{2}{a_5 - a_6}, \\
A_6 &= \frac{1}{a_1 - a_6} + \frac{1}{a_2 - a_6} + \frac{1}{a_3 - a_6} + \frac{1}{a_4 - a_6} + \frac{2}{a_5 - a_6},
\end{align*}
\]

(4)

under the following condition on the poles

\[
6s_4 - 3s_5 a_5 + s_2 a_5^2 + (-3s_3 + 4s_2 a_5 - 3s_1 a_5^2) a_6 + (s_2 - 3s_1 a_5 + 6a_5^2) a_6^2 = 0,
\]

(5)

where \(s_1, \ldots, s_4\) are elementary symmetric polynomials in \(a_1, \ldots, a_4\).

Consideration of the case \(U = 0\) requires that the equation for \(A_1\) has the form \(p_0 A_1^6 + p_1 A_1^4 + p_2 A_1^2 + p_3 A_1 + p_4 = 0\), where \(p_i\) are polynomials in \(a_1, \ldots, a_6\).
Theorem 2. Let $A_k$ be a solution of the fifth degree equation for some fixed values of the poles $a_k$ such that $Q_k \neq 0$. Then $A_k$ and $A_j$ $(k = 2, \ldots, 6)$, evaluated on the basis of this value of $A_k$ and formulas (2), (3), define the solution of the system $(A)$.

Thus, theorems 1 and 2 determine the solution of the Chazy $\mathcal{A}$-system. It should be noted that theorem 1 is a special case of theorem 2. However, under condition (5) the $A_k$ can be determined in the closed form (4).

Solution of system $(B) - (F)$

The solution of the system $(A) - (F)$ for known $A_k$ is reduced to the successive solution of three linear algebraic systems with additional constraints. In the general case, the solution of the systems $(B), (C), (F)$ can be obtained by the Gauss method. Therefore, the system $(B)$ with known $A_k$ is a linear system $A_B B = R_B$ with respect to $B = (B_1, \ldots, B_6)^T$, where matrix $A_B$ has the entries:

$$\begin{align*}
\{A_B\}_{kj} &= \frac{A_k}{2} + \frac{1}{a_k - a_j}, \ j \neq k; \\
\{A_B\}_{kk} &= -\frac{5A_k}{2} - \sum_{i=1}^{6} \frac{1}{a_k - a_i}, \ i \neq k.
\end{align*}$$

In general, the rank of this matrix does not exceed five. The vector $R_B$ depends only on $A_k$, $a_k$ and their derivatives. Under the condition of theorem 1, the matrix $A_B$ can be represented in closed form as well. To do this we need to substitute the values $A_1, \ldots, A_6$ from (4) into the above matrix $A_B$.

With known $A_k$ and $B_k$ from the equation $(D)$ we find that

$$D = -\frac{1}{2} \sum_{j=1}^{6} B_j + \frac{3}{2} a_k A_j,$$

The system $(C)$ with respect to $C = (C_1, \ldots, C_6)^T$ is also linear and can be represented in the form $A_C C = R_C$ with the matrix $A_C$:

$$\begin{align*}
\{A_C\}_{kj} &= \frac{1}{a_k - a_j}, \ j \neq k; \\
\{A_C\}_{kk} &= -2A_k - \sum_{i=1}^{6} \frac{1}{a_k - a_i}, \ i \neq k.
\end{align*}$$

The rank of the matrix $A_C$ also does not exceed five. In this case, the inhomogeneity vector $R_C$ contains $A_k$, $B_k$, $a_k$, their derivatives and the unknown function $E(z)$. Under the condition of theorem 1, the matrix $A_C$ can also be written explicitly. To do this we substitute the values $A_1, \ldots, A_6$ from (4) into the above matrix $A_C$.

The system $(F_4) - (F_6)$ is also linear with respect to $F = (F_1, \ldots, F_6)^T$. In this case, the inhomogeneity matrix $R_F$ contains the unknown function $E(z)$. However, the substitution of $F_k$ into the equations $(F_4) - (F_6)$ allows us to define $E(z)$.

A case of constant poles

The Chazy equation in the case of constant poles $a_k$ has the form

$$y''' = y'y'' \sum_{k=1}^{6} \frac{1}{y - a_k} + y^3 \sum_{k=1}^{6} \frac{A_k}{y - a_k} + y^2 \sum_{k=1}^{6} \frac{B_k}{y - a_k} +$$

$$+ y' \sum_{k=1}^{6} \frac{C_k}{y - a_k} + Dy''' + E y' + \prod_{k=1}^{6} (y - a_k) \sum_{k=1}^{6} \frac{F_k}{y - a_k}.$$

The system $(A)$ remains the same, and the system $(B) - (F)$ is greatly simplified and acquires the form

$$\begin{align*}
(B') &\quad \left(-\frac{5}{2} A_k - \sum_{j=1}^{6} \frac{1}{a_k - a_j}\right) B_j + \sum_{j=1}^{6} \left(\frac{A_k}{2} + \frac{1}{a_k - a_j}\right) B_j = 0, \ k = 1, \ldots, 6 \quad (j \neq k), \\
(C') &\quad \left(-2A_k - \sum_{j=1}^{6} \frac{1}{a_k - a_j}\right) C_k + \sum_{j=1}^{6} \frac{C_j}{a_k - a_j} - B_k^2 + B_k' - B_k D + E = 0, \\
(D') &\quad 2D + \sum_{j=1}^{6} B_j = 0,
\end{align*}$$

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The solution of the \( A \)-system in closed form is given by theorem 1. The solution of the system \( (B^+)^* - (F^+) \) in closed form is obtained in [14]. Using these results, we investigate the integrability of the Chazy equation (6).

Let us consider the case when \( B_k = 0, k = 1, \ldots, 6 \). The following statements are true regarding to the definition of the remaining coefficients of the Chazy equation (6).

**Lemma 2.** If in the Chazy equation (6) \( A_k \) are determined by theorem 1 and \( B_k = 0, k = 1, \ldots, 6 \), then the fulfillment of the Chazy system \( (B^+)^* - (F^+) \) necessarily leads to

\[
D = 0, F_k = 0, k = 1, \ldots, 6.
\]

**Lemma 3.** Let the following conditions be fulfilled in the Chazy equation (6):

A1. The coefficients \( A_k \) and the poles \( a_k \) are defined by theorem 1 and \( B_k, C_k, D, E \) satisfy the Chazy system \( (B^+)^* - (F^+) \).

A2. \( B_k = 0, k = 1, \ldots, 6 \).

A3. The constant poles \( a_k \) satisfy the condition

\[
h_1 := a_1 + a_2 + a_3 + a_4 - a_5 - 3a_6 \neq 0.
\]

Then \( C_6 = C_6, E = E \), where \( C_6, E \) are the arbitrary constants and \( C_1 - C_5 \) are determined by the formulas

\[
C_j = \frac{E}{h_1} (a_j - a_6)^2 + \frac{a_1 - a_6}{h_1} \frac{2a_j - a_6 - a_6}{a_5 - a_6} h_1 - 2 \left( a_j - a_6 \right)^2, \quad j = 1, \ldots, 4.
\]

\[
C_5 = \frac{E}{h_1} (a_5 - a_6)^2 - \frac{a_1 - a_6}{h_1} \frac{2a_5 - 2a_6}{a_5 - a_6} h_1 - 2a_5 - 2a_6.
\]

**Lemma 4.** If the conditions A1, A2 of lemma 3 are fulfilled and \( h_1 = 0 \), then \( C_5 = C_5, E = E \), where \( C_5, E \) are the arbitrary constants and \( C_1 - C_4, C_6 \) are determined by the formulas

\[
C_j = -\frac{E}{2} \frac{(a_j - a_5)^2}{a_5 - a_6}, \quad j = 1, \ldots, 4.
\]

\[
C_6 = \frac{E(a_5 - a_6)}{2}.
\]

The proof of lemmas 2–4 follows directly from the result of the paper [14], where the solution of the Chazy system (6) is given in the case of theorem 1. The above formulas for determining the remaining coefficients of the equation (6) follow from the corresponding formulas in the paper [14] with \( B_k = 0, k = 1, \ldots, 6 \).

The general integral of the equation (6) will be sought in the form

\[
(w')^2 = K_1 P(w) + K_2 Q(w) + R(w),
\]

where \( K_1, K_2 \) are the arbitrary coefficients and \( P(w), Q(w), R(w) \) are the polynomials of \( w \) with constant coefficients not higher than the fourth degree. In this case, the third constant of integration is obtained by separating the variables in the equation (7) and its integration.

It is clear that in the case of the existence of such polynomials \( P(w), Q(w), R(w) \), the solution of (7) and, therefore, the solution of the equation (6) is generally expressed in a rational way in terms of the Weierstrass elliptic function \( \wp(z) \).

Differentiating twice (7) and excluding arbitrary constants \( K_1 \) and \( K_2 \), we find

\[
w'' = w' P'' Q - P Q'' - \frac{P''}{P Q} - \frac{Q''}{P Q'} - \frac{P'' Q'}{P Q'} - \frac{Q'' (P R - P R') - P'' (R Q - R Q')}{2 (P Q - P Q')}.
\]

Hereinafter the primes on the polynomials \( P(w), Q(w), R(w) \) denote the corresponding derivatives with respect to \( w \). Comparing this equation with the equation (6) we have three conditions for the definition of the polynomials \( P(w), Q(w), R(w) \):
\[
\frac{P''Q - PQ''}{P'Q - PQ'} = \sum_{k=1}^{6} \frac{1}{w-a_k}, \\
\frac{P''Q' - P'Q''}{2(P'Q - PQ')} = \sum_{k=1}^{6} \frac{A_k}{w-a_k}, \\
\frac{P(Q'R'' - Q''R') + Q(R'P'' - R''P') + R(P'Q'' - P''Q')}{2(P'Q - PQ')} = E + \sum_{k=1}^{6} \frac{C_k}{w-a_k},
\]

moreover \(B_1 = 0, F_1 = 0, D = 0\).

Without loss of generality in (8) – (10) we consider
\[
P'Q - PQ' = \prod_{k=1}^{6} (w-a_k),
\]

where \(P = \sum_{j=0}^{4} p_j w^j, Q = \sum_{j=0}^{4} q_j w^j, R = \sum_{j=0}^{4} r_j w^j\) and in this case
\[
p_4 = 0, q_4 = 1, q_3 = 0.
\]

Then from the condition (8) which has the form (11) we find
\[
p_1 = -1, p_2 = \frac{\sigma_1}{2}, p_1 = \frac{1}{3}(-q_2 - \sigma_2), \\
p_0 = \frac{1}{4}(-2q_1 + \sigma_3), q_0 = \frac{1}{18}(2q_2^2 + 3q_1\sigma_1 + 2q_2\sigma_2 - 6\sigma_4),
\]

where \(\sigma_k\) here and below are the symmetric polynomials with respect to \(a_1, \ldots, a_6\), and \(q_1, q_2\) must satisfy the following two conditions:
\[
-\frac{1}{4}q_1(-2q_1 + \sigma_3) - \frac{1}{54}(q_2 + \sigma_2)(2q_2^2 + 3q_1\sigma_1 + 2q_2\sigma_2 - 6\sigma_4) - \sigma_6 = 0, \\
q_1q_2 - \frac{q_2q_3}{2} + \frac{1}{18}\sigma_1(2q_2^2 + 3q_1\sigma_1 + 2q_2\sigma_2 - 6\sigma_4) + \sigma_5 = 0.
\]

The condition (9) recorded by the virtue of (11) in the form
\[
P''Q' - P'Q'' = -2(P'Q - PQ') \sum_{k=1}^{6} \frac{A_k}{w-a_k},
\]

under the fulfillment of (12), (13) and \(A_k\) from theorem 1, defines \(q_1, q_2\):
\[
q_1 = \frac{2}{3}\left(a_1a_4 + a_2(a_1 + a_4) + a_3(a_2 + a_3 + a_4)\right)(a_5 + a_6), \\
q_2 = -(a_1 + a_2 + a_3 + a_4)a_5 - (a_1 + a_2 + a_3 + a_4 - 2a_5)a_6.
\]

Note that the values \(q_1, q_2\), defined by (15), satisfy the conditions (14).

The condition (10) by the virtue of (11) takes the form
\[
P(Q'R'' - Q''R') + Q(R'P'' - R''P') + R(P'Q'' - P''Q') = 2(P'Q - PQ') \left( E + \sum_{k=1}^{6} \frac{C_k}{w-a_k} \right).
\]

This condition leads to the determination of the coefficients of the polynomial \(R\). Wherein two cases must be considered corresponding to lemmas 3 and 4.

In the case of lemma 3, that is \(h_1 \neq 0\), the coefficients of the polynomial \(R\) are determined by the relations
\[
r_0 = -\left(2C_0 - C_1a_5\right)(-3a_2a_3a_4 - 3a_1(a_2a_4 + a_2(a_3 + a_4)) + (a_1a_4 + a_2(a_3 + a_4) + a_3(a_2 + a_3 + a_4))a_5 + \\
+ (-3C(a_2a_3a_4 + a_1(a_2a_3 + (a_2 + a_3)a_4)) + 3C(a_1 + a_2 + a_3 + a_4)a_5^2 + \\
+ 2C(a_2a_3 + a_2a_4 + a_3a_4 + 3(a_2 + a_3 + a_4)a_5 - 6a_5^2 + a_1(a_2 + a_3 + a_4 + 3a_5))a_6 + \\
\ldots)
\]
\[+ \left( \mathcal{E}(a_4a_5 + a_5a_3 + a_4) + a_1(a_2 + a_3 + a_4) + 3(4\mathcal{E} + \mathcal{E}(a_1 + a_2 + a_3 + a_4))a_5 - \\
-12\mathcal{E}a_5^2 \right)a_6^5 + 12\mathcal{E}a_5a_6^4) \right) / \left(6h_1(a_5 - a_6)\right)\],
\[r_1 = \left(2\mathcal{E}(a_5 - a_6)\right) a_5(a_4 + a_5(a_3 + a_4) + a_1(a_2 + a_3 + a_4) - 3a_6(a_5 + a_6)) + C(-2a_3a_4 + 3a_5a_6 + \\
+ 3a_4a_5 - 3a_5^2 + 3(a_3 + a_4 - 2a_5)a_6 - 3a_6^2 + a_1(-2a_2 - 2a_3 - 2a_4 + 3(a_5 + a_6)) + \\
+ a_1(-2a_2 - 2a_3 - 2a_4 + 3(a_5 + a_6))) \right) / \left(3h_1(a_5 - a_6)\right),
\]
where \(C = C_0, E = \mathcal{E}\) and \(C_5, \mathcal{E}\) are the arbitrary constants. The rest of the equations of the condition (16) become identity due to (17) and the values of the coefficients \(C_5\) from lemma 3 and the condition of theorem 1.

In the case of lemma 4, that is, \(h_1 = 0\), the coefficients of the polynomial \(R\) are determined by the relations
\[r_0 = \left( a_4(a_4 - a_5)a_5(2C_5 + \mathcal{E}a_5) + (2C_5a_5^2 + 3a_5^2 (2C_5 - \mathcal{E}a_5) - a_5a_6(8C_5 + 3\mathcal{E}a_5))a_6 + (-a_4(6C_5 + \mathcal{E}a_5) + \\
+ (-6C_5 + \mathcal{E}a_5)a_5 + 18\mathcal{E}a_5^2) a_6^2 + 3\mathcal{E}(a_4 - 3a_5)a_6^2 - a_5^2 (3a_4 - a_5 - a_6)(2C_5 + \mathcal{E}a_5 - \mathcal{E}a_6) - \\
a_5(a_4 - a_5 - 3a_6)(3a_4 - a_5 - a_6)(2C_5 + \mathcal{E}a_5 - \mathcal{E}a_6) - a_5^2 (3a_4 - a_5 - a_6)(2C_5 + \mathcal{E}a_5 - \mathcal{E}a_6) - \\
a_6(a_4 + a_5 - 3a_6)(3a_4 + 3a_5 - a_6)(2C_5 + \mathcal{E}a_5 - \mathcal{E}a_6)) \right) / \left(12(a_5 - a_6)^2\right),
\]
\[r_1 = \left( 2C_5a_5^2 + 2C_5a_6^2 + 2C_5a_5^2 + 2C_5a_6^2 + \mathcal{E}a_5a_6 + \mathcal{E}a_5a_6 + \mathcal{E}a_5a_6 + \mathcal{E}a_5a_6 - 2C_5a_6^2 + 5\mathcal{E}a_6^2 - \\
- (4C_5a_1 + \mathcal{E}a_1^2 + 4C_5a_5 + \mathcal{E}a_5^2 + 4C_5a_6 + \mathcal{E}a_6^2 + 4C_5a_6 + \mathcal{E}a_6^2 + \\
+ 2(-2C_5 + \mathcal{E}(a_1 + a_2 + a_3 + a_4))a_4 + 3\mathcal{E}a_5) a_6 + \\
+ (6C_5 + 2\mathcal{E}(a_1 + a_2 + a_3 + a_4) - 5\mathcal{E}a_5)a_6^2 + 3\mathcal{E}a_6^3) / \left(6(a_5 - a_6)^2\right),
\]
where \(C_5 = C_5, E = \mathcal{E}\) and \(C_5, \mathcal{E}\) are the arbitrary constants. The rest of the equations of (16) become identity due to (18) and the values of the coefficients \(C_5\) from lemma 4 and the condition of theorem 1. The above considerations imply the following statement.

**Theorem 3.** If in the Chazy equation (6) \(B_k = 0, k = 1, \ldots, 6\), then under the conditions of the Chazy system \((\mathcal{B}^+) - (\mathcal{F}^+)\) and \(A_k\), defined by theorem 1, the equation (6) is generally integrated in elliptic functions.

**Proof.** If \(B_k = 0\) and \((\mathcal{B}^+) - (\mathcal{F}^+)\) hold then by lemma 1 \(D = 0\) and \(F_k = 0\). By the virtue of theorem 1, we have two cases: \(h_1 \neq 0\) and \(h_1 = 0\) to define \(C_5\). In both cases, the polynomials \(P(w), Q(w), R(w)\) are chosen according to the formulas (12), (13), (15), (17), (18) which proves the statement of the theorem.

Note that lemmas 3 and 4, on which the proof of theorem 3 is based, are given in [15; 16], respectively.

Now consider the following example.

Let \(a_1 = 2, a_2 = -1, a_3 = 2, a_4 = -2, a_5 = 0\). Then from (5) we find \(a_1 = -\frac{8}{5}\), and from (4) we obtain the solution of the system \((\mathcal{A})\):
\[A_1 = -\frac{15}{8}, A_2 = -\frac{4}{3}, A_3 = 0, A_4 = -\frac{3}{4}, A_5 = \frac{37}{12}, A_6 = \frac{7}{8}.\]

In this case \(h_1 = \frac{12}{5}\). Therefore, setting \(B_k = 0, k = 1, \ldots, 6\), from the Chazy system we have \(D = 0\) and \(F_k = 0\), and from lemma 3
\[C_4 = \frac{5}{3}C_6 + \frac{16}{15}\mathcal{E}, C_5 = \frac{1}{12}(-19C_6 + 5\mathcal{E}), C_3 = \frac{5}{12}(C_6 + \mathcal{E}), C_4 = \frac{1}{3}(-4C_6 + 5\mathcal{E}), C_5 = -\frac{1}{3}(8C_6 + 5\mathcal{E}), C_6 = C_6, E = \mathcal{E}.\]
The coefficients $C_6 = C_6$, $E = E$ are remain arbitrary, and the coefficients of the polynomials $P(w)$ and $Q(w)$, respectively, have the following form:

$$
p_4 = 0, \quad p_3 = -1, \quad p_2 = -\frac{4}{5}, \quad p_1 = \frac{7}{5}, \quad p_0 = -\frac{4}{5},
$$

$$
q_4 = 1, \quad q_3 = 0, \quad q_2 = \frac{4}{5}, \quad q_1 = \frac{28}{5}, \quad q_0 = -\frac{16}{5}.
$$

From (17) we find the coefficients of the polynomial $R(w)$:

$$
r_4 = r_3 = 0, \quad r_2 = -E, \quad r_1 = \frac{1}{6}(5C_6 - 7E), \quad r_0 = \frac{2}{3}(C_6 + E).
$$

The general integral of the Chazy equation in this case has the form

$$
(w^4)' = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4,
$$

where

$$
b_0 = \frac{2}{15} \left(6K_1 + 24K_2 - 5C_6 - 5E\right), \quad b_1 = -\frac{7}{5}K_1 - \frac{28}{5}K_2 - \frac{5}{6}C_6 + \frac{7}{6}E,
$$

$$
b_2 = \frac{4}{5}(K_1 - K_2) + E, \quad b_3 = K_1, \quad b_4 = -K_2
$$

and $K_1, K_2$ are the arbitrary constants. The third arbitrary constant appears from the separation of variables in the equation (19) and its integration. Thus, for example, if $K_2 = 0, K_1 \neq 0$, then

$$
w = \alpha \rho(z) + \beta,
$$

where $\alpha = \frac{4}{K_1}$, $\beta = \frac{E}{3K_1} - \frac{4}{15}$ and $\rho(z)$ is the elliptic Weierstrass function satisfying the equation

$$
(\rho')^2 = 4\rho^3 - g_2 \rho - g_3,
$$

$$
g_3 = \frac{242K_1^2 + 125K_1 C_6 - 95K_1 E + 50E^2}{600},
$$

$$
g_3 = \frac{-8176K_1^3 - 500E^3 + 75K_1 E (19E - 25C_6) + 30K_2^2 (100C_6 + 83E)}{108000}.
$$

### Bibliografische ссылки

References


Received 28.04.2021 / revised 01.07.2021 / accepted 01.07.2021.