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О РЕШЕНИЯХ УРАВНЕНИЯ ШАЗИ

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Система Шази определяет необходимые и достаточные условия отсутствия подвижных критических точек у решений дифференциального уравнения третьего порядка, рассмотренного Шази в одной из первых работ по классификации обыкновенных дифференциальных уравнений высших порядков относительно свойства Пенлеве. Решение полной системы Шази в случае постоянных полюсов уже получено. Однако до сих пор вопрос об интегрировании уравнения Шази оставался открытым. В настоящей работе доказывается, что в случае постоянных полюсов при некоторых дополнительных условиях это уравнение интегрируется в эллиптических функциях.

Ключевые слова: уравнение Шази; система Шази; свойство Пенлеве; эллиптические функции.

ON SOLUTIONS OF THE CHAZY EQUATION

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The Chazy system determines the necessary and sufficient conditions for the absence of movable critical points of solutions of the particular third order differential equation that was considered by Chazy in one of the first papers on the classification of higher-order ordinary differential equations with respect to the Painlevé property. The solution of the complete Chazy system in the case of constant poles has been already obtained. However, the question of integrating the Chazy equation remained open until now. In this paper, we prove that in the case of constant poles, under some additional conditions, this equation is integrated in elliptic functions.

Keywords: Chazy equation; Chazy system; Painlevé property; elliptic functions.

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Introduction

It is known that the Painlevé equations appeared as a result of solving the classification problem regarding the Painlevé property for the second order ordinary differential equations [1; 2]. For the equations of higher orders, the Painlevé problem, which consists in determining the conditions for the absence of movable multi-valued singularities for solutions, in the general case remains open. At present, the most complete results have been obtained only for polynomial equations of higher orders. So, for example, equations of the form

$$y''' = P(z, y, y', y''),$$

where $P(z, y, y', y'')$ is a polynomial of y and its derivatives are considered in [3–10]. Some classes of equations of the fourth and higher orders were studied in [11; 12].

One of the first works on the classification of higher-order equations with respect to the Painlevé property was a paper by Chazy [3]. It deals with the Painlevé property of the equation

$$y''' = \sum_{k=1}^{6} \frac{(y' - a_k')(y'' - a_k'')}{y - a_k} + \sum_{k=1}^{6} \frac{A_k(y' - a_k')^3 + B_k(y' - a_k')^2 + C_k(y' - a_k')}{y - a_k} + \\ + Dy'' + Ey' + \prod_{k=1}^{6}(y - a_k)\sum_{k=1}^{6}\frac{F_k}{y - a_k}, \tag{1}$$

where the poles $a_k = a_k(z)$ are finite, distinct and in general are functions of the independent variable z.

The paper [3] also contains the system of 31 algebraic and differential equations

$$(\mathcal{A}) \qquad \sum_{j=1}^{6} A_j = 0,\ \sum_{j=1}^{6} a_j A_j = -6,\ \sum_{j=1}^{6} a_j^2 A_j = -2\sum_{j=1}^{6} a_j,\ 2A_k^2 + \sum_{j=1}^{6}\frac{A_k - A_j}{a_k - a_j} = 0,\ k = 1, \ldots, 6\ (j \neq k),$$

$$(\mathcal{B}) \qquad \sum_{j=1}^{6}(B_k - B_j)\left(-\frac{A_k}{2} - \frac{1}{a_k - a_j}\right) + A_k' - \sum_{j=1}^{6}\frac{a_k' - a_j'}{a_k - a_j}(A_k - 3A_j) - \frac{3}{2}A_k\sum_{i=1}^{6}a_i'A_i = 0,$$

$$(\mathcal{C}) \qquad \left(-2A_kC_k - \sum_{j=1}^{6}\frac{C_k - C_j}{a_k - a_j}\right) + \sum_{j=1}^{6}\frac{3A_j(a_k' - a_j')^2 + (2B_j - B_k)(a_k' - a_j') + a_k'' - a_j''}{a_k - a_j} - \\ - B_k^2 + B_k' - B_kD + E = 0,$$

$$(\mathcal{D}) \qquad 2D + \sum_{j=1}^{6}(B_j - 3a_j'A_j) = 0,$$

$$(\mathcal{F}) \qquad \sum_{j=1}^{6} F_j = 0,\ \sum_{j=1}^{6} a_j F_j = 0,\ \sum_{j=1}^{6} a_j^2 F_j = 0,\ -a_k''' - B_kC_k + C_k' + D(a_k'' - C_k) + Ea_k' + F_k\prod_{j=1}^{6}(a_k - a_j) + \\ + \sum_{j=1}^{6}\frac{A_j(a_k' - a_j')^3 + B_j(a_k' - a_j')^2 - (C_k - C_j)(a_k' - a_j') + (a_k' - a_j')(a_k'' - a_j'')}{a_k - a_j} = 0,$$

for 26 unknown functions $A_k = A_k(z), B_k = B_k(z), C_k = C_k(z), D = D(z), E = E(z), F_k = F_k(z)$. In [3, p. 367–369] Chazy claimed that the solution of this system determines necessary and sufficient conditions for the absence of movable critical points of solutions of the equation (1). In this case, the poles a_k are the parameters of the system. The equations of the system $(\mathcal{A})-(\mathcal{F})$ we denote below by $(\mathcal{A}_1), \ldots, (\mathcal{A}_9), (\mathcal{B}_1), \ldots, (\mathcal{B}_6), (\mathcal{C}_1), \ldots, (\mathcal{C}_6), (\mathcal{D}), (\mathcal{F}_1), \ldots, (\mathcal{F}_9)$.

The solution of the Chazy $\mathcal{A}$-system was obtained in [13] and the solution of the complete Chazy system in the case of constant poles a_k in expanded form is given in [14]. However, the question of integrating the Chazy equation (1) remained open until now. In this paper, we prove that in the case of constant poles, under some additional conditions, the equation (1) is integrated in elliptic functions.

Let us briefly summarise some results from [13; 14] which we need to obtain the main result stated in theorem 3.

Solution of system $(\mathcal{A})$

First, let us note that the successive elimination of the variables A_k allows us uniquely express A_6, A_5, A_4, A_3, A_2 in terms of A_1. Indeed from the system $(\mathcal{A})$ we successively obtain

$$A_6 = -A_5 - A_4 - A_3 - A_2 - A_1,\ A_5 = -\frac{6}{a_{56}} - \frac{a_{46}}{a_{56}} A_4 - \frac{a_{36}}{a_{56}} A_3 - \frac{a_{26}}{a_{56}} A_2 - \frac{a_{16}}{a_{56}} A_1,$$

$$A_4 = \frac{2\left(a_{41} + a_{42} + a_{43} - a_{45} - a_{46}\right)}{a_{45}a_{46}} - \frac{a_{35}a_{36}}{a_{45}a_{46}} A_3 - \frac{a_{25}a_{26}}{a_{45}a_{46}} A_2 - \frac{a_{15}a_{16}}{a_{45}a_{46}} A_1, \quad (2)$$

$$A_3 = \frac{2a_{13}\left(-a_{12} - a_{13} + 2a_{14} + 2a_{15} + 2a_{16}\right)}{a_{34}a_{35}a_{36}} - \frac{a_{13}a_{24}a_{25}a_{26}}{a_{12}a_{34}a_{35}a_{36}} A_2 +$$

$$+ \frac{a_{13}a_{14}a_{15}a_{16}}{a_{34}a_{35}a_{36}}\left(\frac{1}{a_{12}} + \frac{1}{a_{13}} + \frac{2}{a_{14}} + \frac{2}{a_{15}} + \frac{2}{a_{16}}\right)A_1 + \frac{2a_{13}a_{14}a_{15}a_{16}}{a_{34}a_{35}a_{36}} A_1^2,$$

where a_{ij} denote the non-zero pole differences $a_i - a_j$. Let's also note the structure of A_2:

$$A_2 = \frac{n_0 A_1^4 + n_1 A_1^3 + n_2 A_1^2 + n_3 A_1 + n_4}{\left(a_2 - a_4\right)\left(a_2 - a_5\right)\left(a_2 - a_6\right)Q_2}, \quad (3)$$

where $Q_2 = d_0 A_1^2 + d_1 A_1 + d_2$, and the coefficients $n_0, \ldots, n_4$ и d_0, d_1, d_2 are determined through $a_1, \ldots, a_6$.

After the substitution of the expressions thus obtained into the system $(\mathcal{A})$, the first four equations of this system become identities, and the equations $(\mathcal{A}_5)-(\mathcal{A}_9)$ acquire the form

$$\frac{a_{21}^2 u_{45}u_{46}u_{56}U}{a_{24}^2 a_{25}^2 a_{26}^2 Q_2^2} = 0,\ \frac{a_{31}^2 u_{45}u_{46}u_{56}U}{a_{34}^2 a_{35}^2 a_{36}^2 Q_2^2} = 0,\ \frac{a_{41}^2 u_{56}U_4 U}{a_{42}^2 a_{43}^2 a_{45}^2 a_{46}^2 Q_2^2} = 0,$$

$$\frac{a_{51}^2 u_{46}U_5 U}{a_{52}^2 a_{53}^2 a_{54}^2 a_{56}^2 Q_2^2} = 0,\ \frac{a_{61}^2 u_{45}U_6 U}{a_{62}^2 a_{63}^2 a_{64}^2 a_{65}^2 Q_2^2} = 0,$$

respectively, where $a_{ij} = a_i - a_j$, $u_{ij} = 2a_1 - a_i - a_j + \left(a_1 - a_i\right)\left(a_1 - a_j\right)A_1$; U and U_4, U_5, U_6 are polynomials of the fifth and second degrees in A_1, respectively.

Next two theorems from [14] follow from the successive considering of the two cases ($u_{ij} = 0$ and $U = 0$).

Lemma 1. *The system* $(\mathcal{A})$ *admits the symmetry* $\left(A_k, a_k\right) \leftrightarrow \left(A_j, a_j\right)$, $j, k = 1, \ldots, 6$.

This lemma shows that the permutation of arbitrary components $\left(A_k, a_k\right)$ of the solution of the system $(\mathcal{A})$ with arbitrary components $\left(A_j, a_j\right)$ leads to the solution of the system $(\mathcal{A})$.

Theorem 1. *The system* $(\mathcal{A})$ *has the solution*

$$A_j = \frac{1}{a_5 - a_j} + \frac{1}{a_6 - a_j},\ j = 1, \ldots, 4,$$

$$A_5 = \frac{1}{a_1 - a_5} + \frac{1}{a_2 - a_5} + \frac{1}{a_3 - a_5} + \frac{1}{a_4 - a_5} + \frac{2}{a_5 - a_6}, \quad (4)$$

$$A_6 = \frac{1}{a_1 - a_6} + \frac{1}{a_2 - a_6} + \frac{1}{a_3 - a_6} + \frac{1}{a_4 - a_6} + \frac{2}{a_6 - a_5}$$

under the following condition on the poles

$$6s_4 - 3s_3 a_5 + s_2 a_5^2 + \left(-3s_3 + 4s_2 a_5 - 3s_1 a_5^2\right)a_6 + \left(s_2 - 3s_1 a_5 + 6a_5^2\right)a_6^2 = 0, \quad (5)$$

where $s_1, \ldots, s_4$ *are elementary symmetric polynomials in* $a_1, \ldots, a_4$.

Consideration of the case $U = 0$ requires that the equation for A_1 has the form $p_0 A_1^5 + p_1 A_1^4 + p_2 A_1^3 + p_3 A_1^2 + p_4 A_1 + p_5 = 0$, where p_i are polynomials in $a_1, \ldots, a_6$.

Theorem 2. *Let A_1 be a solution of the fifth degree equation for some fixed values of the poles a_k such that $Q_2 \neq 0$. Then A_1 and A_k ($k = 2, \ldots, 6$), evaluated on the basis of this value of A_1 and formulas (2), (3), define the solution of the system $(\mathcal{A})$.*

Thus, theorems 1 and 2 determine the solution of the Chazy $\mathcal{A}$-system. It should be noted that theorem 1 is a special case of theorem 2. However, under condition (5) the A_k can be determined in the closed form (4).

Solution of system $(\mathcal{B})-(\mathcal{F})$

The solution of the system $(\mathcal{A})-(\mathcal{F})$ for known A_k is reduced to the successive solution of three linear algebraic systems with additional constraints. In the general case, the solution of the systems $(\mathcal{B})$, $(\mathcal{C})$, $(\mathcal{F})$ can be obtained by the Gauss method. Therefore, the system $(\mathcal{B})$ with known A_k is a linear system $A_B B = R_B$ with respect to $B = (B_1, \ldots, B_6)^T$, where matrix A_B has the entries:

$$\{A_B\}_{kj} = \frac{A_k}{2} + \frac{1}{a_k - a_j},\ j \neq k,\ \{A_B\}_{kk} = -\frac{5A_k}{2} - \sum_{i=1}^{6} \frac{1}{a_k - a_i},\ i \neq k.$$

In general, the rank of this matrix does not exceed five. The vector R_B depends only on A_k, a_k and their derivatives. Under the condition of theorem 1, the matrix A_B can be represented in closed form as well. To do this we need to substitute the values $A_1, \ldots, A_6$ from (4) into the above matrix A_B.

With known A_k and B_k from the equation $(\mathcal{D})$ we find that

$$D = -\frac{1}{2}\sum_{j=1}^{6} B_j + \frac{3}{2} a_j' A_j.$$

The system $(\mathcal{C})$ with respect to $C = (C_1, \ldots, C_6)^T$ is also linear and can be represented in the form $A_C C = R_C$ with the matrix A_C:

$$\{A_C\}_{kj} = \frac{1}{a_k - a_j},\ j \neq k;\ \{A_C\}_{kk} = -2A_k - \sum_{i=1}^{6} \frac{1}{a_k - a_i},\ i \neq k.$$

The rank of the matrix A_C also does not exceed five. In this case, the inhomogeneity vector R_C contains A_k, B_k, a_k, their derivatives and the unknown function $E(z)$. Under the condition of theorem 1, the matrix A_C can also be written explicitly. To do this we substitute the values $A_1, \ldots, A_6$ from (4) into the above matrix A_C.

The system $(\mathcal{F}_4)-(\mathcal{F}_9)$ is also linear with respect to $F = (F_1, \ldots, F_6)^T$. In this case, the inhomogeneity matrix R_F contains the unknown function $E(z)$. However, the substitution of F_k into the equations $(\mathcal{F}_1)-(\mathcal{F}_3)$ allows us to define $E(z)$.

A case of constant poles

The Chazy equation in the case of constant poles a_k has the form

$$y''' = y'y''\sum_{k=1}^{6}\frac{1}{y - a_k} + y'^3\sum_{k=1}^{6}\frac{A_k}{y - a_k} + y'^2\sum_{k=1}^{6}\frac{B_k}{y - a_k} + \\ + y'\sum_{k=1}^{6}\frac{C_k}{y - a_k} + Dy'' + Ey' + \prod_{k=1}^{6}(y - a_k)\sum_{k=1}^{6}\frac{F_k}{y - a_k}. \tag{6}$$

The system $(\mathcal{A})$ remains the same, and the system $(\mathcal{B})-(\mathcal{F})$ is greatly simplified and acquires the form

$$(\mathcal{B}^*) \qquad \left(-\frac{5}{2}A_k - \sum_{j=1}^{6}\frac{1}{a_k - a_j}\right)B_k + \sum_{j=1}^{6}\left(\frac{A_k}{2} + \frac{1}{a_k - a_j}\right)B_j = 0,\ k = 1, \ldots, 6 \quad (j \neq k),$$

$$(\mathcal{C}^*) \qquad \left(-2A_k - \sum_{j=1}^{6}\frac{1}{a_k - a_j}\right)C_k + \sum_{j=1}^{6}\frac{C_j}{a_k - a_j} - B_k^2 + B_k' - B_k D + E = 0,$$

$$(\mathcal{D}^*) \qquad 2D + \sum_{j=1}^{6} B_j = 0,$$

$$(\mathcal{F}^*)\qquad \sum_{j=1}^{6} F_j = 0,\ \sum_{j=1}^{6} a_j F_j = 0,\ \sum_{j=1}^{6} a_j^2 F_j = 0,\ -B_k C_k - DC_k + C_k' + F_k \prod_{j=1}^{6}\left(a_k - a_j\right) = 0.$$

The solution of the $\mathcal{A}$-system in closed form is given by theorem 1. The solution of the system $\left(\mathcal{B}^*\right)-\left(\mathcal{F}^*\right)$ in closed form is obtained in [14]. Using these results, we investigate the integrability of the Chazy equation (6).

Let us consider the case when $B_k = 0$, $k = 1, \ldots, 6$. The following statements are true regarding to the definition of the remaining coefficients of the Chazy equation (6).

Lemma 2. *If in the Chazy equation (6) A_k are determined by theorem 1 and $B_k = 0$, $k = 1, \ldots, 6$, then the fulfillment of the Chazy system $\left(\mathcal{B}^*\right)-\left(\mathcal{F}^*\right)$ necessarily leads to*

$$D = 0,\ F_k = 0,\ k = 1, \ldots, 6.$$

Lemma 3. *Let the following conditions be fulfilled in the Chazy equation (6):*

A1. *The coefficients A_k and the poles a_k are defined by theorem 1 and B_k, C_k, D, E satisfy the Chazy system $\left(\mathcal{B}^*\right)-\left(\mathcal{F}^*\right)$.*

A2. $B_k = 0$, $k = 1, \ldots, 6$.

A3. *The constant poles a_k satisfy the condition*

$$h_1 := a_1 + a_2 + a_3 + a_4 - a_5 - 3a_6 \neq 0.$$

Then $C_6 = \mathcal{C}_6$, $E = \mathcal{E}$, where $\mathcal{C}_6$, $\mathcal{E}$ are the arbitrary constants and $C_1 - C_5$ are determined by the formulas

$$C_j = \frac{\mathcal{E}}{h_1}\left(a_j - a_6\right)^2 + \mathcal{C}_6 \frac{\left(2a_j - a_5 - a_6\right)h_1 - 2\left(a_j - a_6\right)^2}{\left(a_5 - a_6\right)h_1},\ j = 1, \ldots, 4,$$

$$C_5 = -\frac{\mathcal{E}}{h_1}\left(a_5 - a_6\right)^2 - \mathcal{C}_6 \frac{h_1 - 2a_5 + 2a_6}{h_1}.$$

Lemma 4. *If the conditions* **A1**, **A2** *of lemma 3 are fulfilled and $h_1 = 0$, then $C_5 = \mathcal{C}_5$, $E = \mathcal{E}$, where $\mathcal{C}_5$, $\mathcal{E}$ are the arbitrary constants and $C_1 - C_4$, C_6 are determined by the formulas*

$$C_j = -\mathcal{E}\frac{\left(a_j - a_5\right)^2}{2\left(a_5 - a_6\right)} - \mathcal{C}_5 \frac{\left(a_j - a_6\right)^2}{\left(a_5 - a_6\right)^2},\ j = 1, \ldots, 4,$$

$$C_6 = \frac{\mathcal{E}\left(a_5 - a_6\right)}{2}.$$

The proof of lemmas 2–4 follows directly from the result of the paper [14], where the solution of the Chazy system (6) is given in the case of theorem 1. The above formulas for determining the remaining coefficients of the equation (6) follow from the corresponding formulas in the paper [14] with $B_k = 0$, $k = 1, \ldots, 6$.

The general integral of the equation (6) will be sought in the form

$$\left(w'\right)^2 = K_1 P(w) + K_2 Q(w) + R(w), \tag{7}$$

where K_1, K_2 are the arbitrary coefficients and $P(w)$, $Q(w)$, $R(w)$ are the polynomials of w with constant coefficients not higher than the fourth degree. In this case, the third constant of integration is obtained by separating the variables in the equation (7) and its integration.

It is clear that in the case of the existence of such polynomials $P(w)$, $Q(w)$, $R(w)$, the solution of (7) and, therefore, the solution of the equation (6) is generally expressed in a rational way in terms of the Weierstrass elliptic function $\wp(z)$.

Differentiating twice (7) and excluding arbitrary constants K_1 and K_2, we find

$$w''' = w'w''\frac{P''Q - PQ''}{P'Q - PQ'} - w'^3\frac{P''Q' - P'Q''}{2\left(P'Q - PQ'\right)} + w'\frac{Q''\left(P'R - PR'\right) - P''\left(R'Q - RQ'\right)}{2\left(P'Q - PQ'\right)}.$$

Hereinafter the primes on the polynomials $P(w)$, $Q(w)$, $R(w)$ denote the corresponding derivatives with respect to w. Comparing this equation with the equation (6) we have three conditions for the definition of the polynomials $P(w)$, $Q(w)$, $R(w)$:

$$\frac{P''Q-PQ''}{P'Q-PQ'}=\sum_{k=1}^{6}\frac{1}{w-a_k}, \tag{8}$$

$$-\frac{P''Q'-P'Q''}{2(P'Q-PQ')}=\sum_{k=1}^{6}\frac{A_k}{w-a_k}, \tag{9}$$

$$\frac{P(Q'R''-Q''R')+Q(R'P''-R''P')+R(P'Q''-P''Q')}{2(P'Q-PQ')}=E+\sum_{k=1}^{6}\frac{C_k}{w-a_k}, \tag{10}$$

moreover $B_k=0$, $F_k=0$, $D=0$.

Without loss of generality in (8)–(10) we consider

$$P'Q-PQ'=\prod_{k=1}^{6}(w-a_k), \tag{11}$$

where $P=\sum_{j=0}^{4}p_jw^j,\ Q=\sum_{j=0}^{4}q_jw^j,\ R=\sum_{j=0}^{4}r_jw^j$ and in this case

$$p_4=0,\ q_4=1,\ q_3=0. \tag{12}$$

Then from the condition (8) which has the form (11) we find

$$\begin{gathered}p_3=-1,\ p_2=\frac{\sigma_1}{2},\ p_1=\frac{1}{3}(-q_2-\sigma_2),\\ p_0=\frac{1}{4}(-2q_1+\sigma_3),\ q_0=\frac{1}{18}\left(2q_2^2+3q_1\sigma_1+2q_2\sigma_2-6\sigma_4\right),\end{gathered} \tag{13}$$

where σ_k here and below are the symmetric polynomials with respect to $a_1, \ldots, a_6$, and q_1, q_2 must satisfy the following two conditions:

$$\begin{gathered}-\frac{1}{4}q_1(-2q_1+\sigma_3)-\frac{1}{54}(q_2+\sigma_2)\left(2q_2^2+3q_1\sigma_1+2q_2\sigma_2-6\sigma_4\right)-\sigma_6=0,\\ q_1q_2-\frac{q_2q_3}{2}+\frac{1}{18}\sigma_1\left(2q_2^2+3q_1\sigma_1+2q_2\sigma_2-6\sigma_4\right)+\sigma_5=0.\end{gathered} \tag{14}$$

The condition (9) recorded by the virtue of (11) in the form

$$P''Q'-P'Q''=-2(P'Q-PQ')\sum_{k=1}^{6}\frac{A_k}{w-a_k},$$

under the fulfillment of (12), (13) and A_k from theorem 1, defines q_1, q_2:

$$\begin{gathered}q_1=\frac{2}{3}\left(a_3a_4+a_2(a_3+a_4)+a_1(a_2+a_3+a_4)\right)(a_5+a_6),\\ q_2=-(a_1+a_2+a_3+a_4)a_5-(a_1+a_2+a_3+a_4-2a_5)a_6.\end{gathered} \tag{15}$$

Note that the values q_1, q_2, defined by (15), satisfy the conditions (14).

The condition (10) by the virtue of (11) takes the form

$$P(Q'R''-Q''R')+Q(R'P''-R''P')+R(P'Q''-P''Q')=2(P'Q-PQ')\left(E+\sum_{k=1}^{6}\frac{C_k}{w-a_k}\right). \tag{16}$$

This condition leads to the determination of the coefficients of the polynomial R. Wherein two cases must be considered corresponding to lemmas 3 and 4.

In the case of lemma 3, that is $h_1\neq 0$, the coefficients of the polynomial R are determined by the relations

$$\begin{gathered}r_0=-\Big(\big(2\mathcal{C}_6-\mathcal{E}a_5\big)\big(-3a_2a_3a_4-3a_1\big(a_3a_4+a_2(a_3+a_4)\big)+\big(a_3a_4+a_2(a_3+a_4)+a_1(a_2+a_3+a_4)\big)a_5\big)+\\ +\Big(-3\mathcal{E}\big(a_2a_3a_4+a_1\big(a_2a_3+(a_2+a_3)a_4\big)\big)+3\mathcal{E}(a_1+a_2+a_3+a_4)a_5^2+\\ +2\mathcal{C}\big(a_2a_3+a_2a_4+a_3a_4+3(a_2+a_3+a_4)a_5-6a_5^2+a_1(a_2+a_3+a_4+3a_5)\big)\Big)a_6+\end{gathered}$$

$$+\Big(\mathcal{E}\big(a_3a_4+a_2(a_3+a_4)+a_1(a_2+a_3+a_4)\big)-3\big(4\mathcal{C}+\mathcal{E}(a_1+a_2+a_3+a_4)\big)a_5-$$
$$-12\mathcal{E}a_5^2\Big)a_6^2+12\mathcal{E}a_5a_6^3\Big)/\big(6h_1(a_5-a_6)\big),$$
$$r_1=\Big(2\Big(\mathcal{E}(a_5-a_6)\big(a_3a_4+a_2(a_3+a_4)+a_1(a_2+a_3+a_4)-3a_6(a_5+a_6)\big)+\mathcal{C}\Big(-2a_3a_4+3a_3a_5+$$
$$+3a_4a_5-3a_5^2+3(a_3+a_4-2a_5)a_6-3a_6^2+a_2\big(-2a_3-2a_4+3(a_5+a_6)\big)+ \qquad (17)$$
$$+a_1\big(-2a_2-2a_3-2a_4+3(a_5+a_6)\big)\Big)\Big)\Big)/\big(3h_1(a_5-a_6)\big),$$
$$r_2=-\mathcal{E},\ r_3=0,\ r_4=0,$$

where $C_6=\mathcal{C}_6$, $E=\mathcal{E}$ and $\mathcal{C}_6$, $\mathcal{E}$ are the arbitrary constants. The rest of the equations of the condition (16) become identity due to (17) and the values of the coefficients C_k from lemma 3 and the condition of theorem 1.

In the case of lemma 4, that is $h_1=0$, the coefficients of the polynomial R are determined by the relations

$$r_0=\Big(a_4(a_4-a_5)a_5(2\mathcal{C}_5+\mathcal{E}a_5)+\big(2\mathcal{C}_5a_4^2+3a_5^2(2\mathcal{C}_5-3\mathcal{E}a_5)-a_4a_5(8\mathcal{C}_5+3\mathcal{E}a_5)\big)a_6+\big(-a_4(6\mathcal{C}_5+\mathcal{E}a_4)+$$
$$+(-6\mathcal{C}_5+\mathcal{E}a_4)a_5+18\mathcal{E}a_5^2\big)a_6^2+3\mathcal{E}(a_4-3a_5)a_6^3-a_3^2(3a_4-a_5-a_6)(2\mathcal{C}_5+\mathcal{E}a_5-\mathcal{E}a_6)-$$
$$-a_3(a_4-a_5-3a_6)(3a_4-a_5-a_6)(2\mathcal{C}_5+\mathcal{E}a_5-\mathcal{E}a_6)-a_1^2(3a_3+3a_4-a_5-a_6)(2\mathcal{C}_5+\mathcal{E}a_5-\mathcal{E}a_6)-$$
$$-a_1(a_3+a_4-a_5-3a_6)(3a_3+3a_4-a_5-a_6)\big(2\mathcal{C}_5+\mathcal{E}a_5-\mathcal{E}a_6\big)\Big)/\Big(12(a_5-a_6)^2\Big),$$
$$r_1=\Big(2\mathcal{C}_5a_1^2+2\mathcal{C}_5a_2^2+2\mathcal{C}_5a_3^2+2\mathcal{C}_5a_4^2+\mathcal{E}a_1^2a_5+\mathcal{E}a_2^2a_5+\mathcal{E}a_3^2a_5+\mathcal{E}a_4^2a_5-2\mathcal{C}_5a_5^2+5\mathcal{E}a_5^3- \qquad (18)$$
$$-\Big(4\mathcal{C}_5a_1+\mathcal{E}a_1^2+4\mathcal{C}_5a_2+\mathcal{E}a_2^2+4\mathcal{C}_5a_3+\mathcal{E}a_3^2+4\mathcal{C}_5a_4+\mathcal{E}a_4^2+$$
$$+2\big(-2\mathcal{C}_5+\mathcal{E}(a_1+a_2+a_3+a_4)\big)a_5+3\mathcal{E}a_5^2\Big)a_6+$$
$$+\Big(6\mathcal{C}_5+2\mathcal{E}(a_1+a_2+a_3+a_4)-5\mathcal{E}a_5\Big)a_6^2+3\mathcal{E}a_6^3\Big)/\Big(6(a_5-a_6)^2\Big),$$
$$r_2=-\mathcal{E},\ r_3=0,\ r_4=0,$$

where $C_5=\mathcal{C}_5$, $E=\mathcal{E}$ and $\mathcal{C}_5$, $\mathcal{E}$ are the arbitrary constants. The rest of the equations of (16) become identity due to (18) and the values of the coefficients C_k from lemma 4 and the condition of theorem 1. The above considerations imply the following statement.

Theorem 3. *If in the Chazy equation (6) $B_k=0$, $k=1,\ldots,6$, then under the conditions of the Chazy system $(\mathcal{B}^*)-(\mathcal{F}^*)$ and A_k, defined by theorem 1, the equation (6) is generally integrated in elliptic functions.*

P r o o f. If $B_k=0$ and $(\mathcal{B}^*)-(\mathcal{F}^*)$ hold then by lemma 1 $D=0$ and $F_k=0$. By the virtue of theorem 1, we have two cases: $h_1\neq0$ and $h_1=0$ to define C_k. In both cases, the polynomials $P(w)$, $Q(w)$, $R(w)$ are chosen according to the formulas (12), (13), (15), (17), (18) which proves the statement of the theorem.

Note that lemmas 3 and 4, on which the proof of theorem 3 is based, are given in [15; 16], respectively.

Now consider the following example.

Let $a_2=1$, $a_3=-1$, $a_4=2$, $a_5=-2$, $a_6=0$. Then from (5) we find $a_1=-\frac{8}{5}$, and from (4) we obtain the solution of the system $(\mathcal{A})$:

$$A_1=-\frac{15}{8},\ A_2=-\frac{4}{3},\ A_3=0,\ A_4=-\frac{3}{4},\ A_5=\frac{37}{12},\ A_6=\frac{7}{8}.$$

In this case $h_1=\frac{12}{5}$. Therefore, setting $B_k=0$, $k=1,\ldots,6$, from the Chazy system we have $D=0$ and $F_k=0$, and from lemma 3

$$C_1=\frac{5}{3}\mathcal{C}_6+\frac{16}{15}\mathcal{E},\ C_2=\frac{1}{12}(-19\mathcal{C}_6+5\mathcal{E}),\ C_3=\frac{5}{12}(\mathcal{C}_6+\mathcal{E}),\ C_4=\frac{1}{3}(-4\mathcal{C}_6+5\mathcal{E}),$$
$$C_5=-\frac{1}{3}(8\mathcal{C}_6+5\mathcal{E}),\ C_6=\mathcal{C}_6,\ E=\mathcal{E}.$$

The coefficients $C_6 = \mathcal{C}_6$, $E = \mathcal{E}$ are remain arbitrary, and the coefficients of the polynomials $P(w)$ and $Q(w)$, respectively, have the following form:

$$p_4 = 0,\ p_3 = -1,\ p_2 = -\frac{4}{5},\ p_1 = \frac{7}{5},\ p_0 = -\frac{4}{5},$$

$$q_4 = 1,\ q_3 = 0,\ q_2 = \frac{4}{5},\ q_1 = \frac{28}{5},\ q_0 = -\frac{16}{5}.$$

From (17) we find the coefficients of the polynomial $R(w)$:

$$r_4 = r_3 = 0,\ r_2 = -\mathcal{E},\ r_1 = \frac{1}{6}(5\mathcal{C}_6 - 7\mathcal{E}),\ r_0 = \frac{2}{3}(\mathcal{C}_6 + \mathcal{E}).$$

The general integral of the Chazy equation in this case has the form

$$(w')^2 = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4, \quad (19)$$

where

$$b_0 = \frac{2}{15}(6K_1 + 24K_2 - 5\mathcal{C}_6 - 5\mathcal{E}),\ b_1 = -\frac{7}{5}K_1 - \frac{28}{5}K_2 - \frac{5}{6}\mathcal{C}_6 + \frac{7}{6}\mathcal{E},$$

$$b_2 = \frac{4}{5}(K_1 - K_2) + \mathcal{E},\ b_3 = K_1,\ b_4 = -K_2$$

and K_1, K_2 are the arbitrary constants. The third arbitrary constant appears from the separation of variables in the equation (19) and its integration. Thus, for example, if $K_2 = 0$, $K_1 \neq 0$, then

$$w = \alpha\rho(z) + \beta,$$

where $\alpha = \frac{4}{K_1}$, $\beta = -\frac{\mathcal{E}}{3K_1} - \frac{4}{15}$ and $\rho(z)$ is the elliptic Weierstrass function satisfying the equation

$$(\rho')^2 = 4\rho^3 - g_2\rho - g_3,$$

$$g_2 = \frac{242K_1^2 + 125K_1\mathcal{C}_6 - 95K_1\mathcal{E} + 50\mathcal{E}^2}{600},$$

$$g_3 = \frac{-8176K_1^3 - 500\mathcal{E}^3 + 75K_1\mathcal{E}(19\mathcal{E} - 25\mathcal{C}_6) + 30K_1^2(100\mathcal{C}_6 + 83\mathcal{E})}{108\,000}.$$

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