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# МАТЕМАТИЧЕСКАЯ ЛОГИКА, АЛГЕБРА И ТЕОРИЯ ЧИСЕЛ

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## MATHEMATICAL LOGIC, ALGEBRA AND NUMBER THEORY

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### КОНЕЧНЫЕ ГРУППЫ С ЗАДАНЫМИ СИСТЕМАМИ ОБОБЩЕННЫХ $\sigma$ -ПЕРЕСТАНОВОЧНЫХ ПОДГРУПП

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Пусть  $\sigma = \{\sigma_i | i \in I\}$  – разбиение множества всех простых чисел  $\mathbb{P}$ , а  $G$  – конечная группа. Множество  $\mathcal{H}$  подгрупп группы  $G$  называется *полным холловым  $\sigma$ -множеством* группы  $G$ , если каждый член  $\neq 1$  из  $\mathcal{H}$  является холловой  $\sigma_i$ -подгруппой группы  $G$  для некоторого  $i \in I$  и  $\mathcal{H}$  содержит ровно одну холлову  $\sigma_i$ -подгруппу группы  $G$  для всех  $i$  таких, что  $\sigma_i \cap \pi(G) \neq \emptyset$ . Группа считается  *$\sigma$ -примарной*, если она есть конечная  $\sigma_i$ -группа для некоторого  $i$ . Подгруппа  $A$  группы  $G$  называется  *$\sigma$ -перестановочной* в  $G$ , если  $G$  содержит полное холлово  $\sigma$ -множество  $\mathcal{H}$  такое, что  $AH^x = H^xA$  для любого  $H \in \mathcal{H}$  и любого  $x \in G$ ;  *$\sigma$ -субнормальной* в  $G$ , если существует подгруппа цепи  $A = A_0 \leq A_1 \leq \dots \leq A_t = G$  такая, что либо  $A_{i-1} \trianglelefteq A_i$ , либо  $A_i / (A_i - 1)_{A_i}$  является  $\sigma$ -примарной для всех  $i = 1, \dots, t$ ;  *$\mathcal{U}$ -нормальной* в  $G$ , если каждый главный фактор группы  $G$  между  $A_G$  и  $A^G$  циклический. Мы говорим, что подгруппа  $H$  группы  $G$  является: (i) *частично  $\sigma$ -перестановочной* в  $G$ , если существуют  $\mathcal{U}$ -нормальная подгруппа  $A$  и  $\sigma$ -перестановочная подгруппа  $B$  из  $G$  такие, что  $H = \langle A, B \rangle$ ; (ii)  *$(\mathcal{U}, \sigma)$ -вложенной* в  $G$ , если существуют частично  $\sigma$ -перестановочная подгруппа  $S$  и  $\sigma$ -субнормальная подгруппа  $T$  из  $G$  такие, что  $G = HT$

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и  $H \cap T \leq S \leq H$ . Мы изучаем  $G$ , предполагая, что некоторые подгруппы группы  $G$  являются частично  $\sigma$ -перестановочными или  $(\mathfrak{U}, \sigma)$ -вложенными в  $G$ . Некоторые известные результаты обобщены.

**Ключевые слова:** конечная группа;  $\sigma$ -разрешимые группы;  $\sigma$ -нильпотентная группа; частично  $\sigma$ -перестановочная подгруппа;  $(\mathfrak{U}, \sigma)$ -вложенная подгруппа;  $\mathfrak{U}$ -нормальная подгруппа.

## FINITE GROUPS WITH GIVEN SYSTEMS OF GENERALISED $\sigma$ -PERMUTABLE SUBGROUPS

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Let  $\sigma = \{\sigma_i | i \in I\}$  be a partition of the set of all primes  $\mathbb{P}$  and  $G$  be a finite group. A set  $\mathcal{H}$  of subgroups of  $G$  is said to be a *complete Hall  $\sigma$ -set* of  $G$  if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $i \in I$  and  $\mathcal{H}$  contains exactly one Hall  $\sigma_i$ -subgroup of  $G$  for every  $i$  such that  $\sigma_i \cap \pi(G) \neq \emptyset$ . A group is said to be  *$\sigma$ -primary* if it is a finite  $\sigma_i$ -group for some  $i$ . A subgroup  $A$  of  $G$  is said to be:  *$\sigma$ -permutable* in  $G$  if  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  such that  $AH^x = H^xA$  for all  $H \in \mathcal{H}$  and all  $x \in G$ ;  *$\sigma$ -subnormal* in  $G$  if there is a subgroup chain  $A = A_0 \leq A_1 \leq \dots \leq A_t = G$  such that either  $A_{i-1} \trianglelefteq A_i$  or  $A_i / (A_i - 1)_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, t$ ;  *$\mathfrak{U}$ -normal* in  $G$  if every chief factor of  $G$  between  $A_G$  and  $A^G$  is cyclic. We say that a subgroup  $H$  of  $G$  is: (i) *partially  $\sigma$ -permutable* in  $G$  if there are a  $\mathfrak{U}$ -normal subgroup  $A$  and a  $\sigma$ -permutable subgroup  $B$  of  $G$  such that  $H = \langle A, B \rangle$ ; (ii)  *$(\mathfrak{U}, \sigma)$ -embedded* in  $G$  if there are a partially  $\sigma$ -permutable subgroup  $S$  and a  $\sigma$ -subnormal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq S \leq H$ . We study  $G$  assuming that some subgroups of  $G$  are partially  $\sigma$ -permutable or  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ . Some known results are generalised.

**Keywords:** finite group;  $\sigma$ -soluble groups;  $\sigma$ -nilpotent group; partially  $\sigma$ -permutable subgroup;  $(\mathfrak{U}, \sigma)$ -embedded subgroup;  $\mathfrak{U}$ -normal subgroup.

### Introduction

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. Moreover,  $\mathbb{P}$  is the set of all primes,  $\pi \subseteq \mathbb{P}$  and  $\pi' = \mathbb{P} \setminus \pi$ . If  $n$  is an integer, the symbol  $\pi(n)$  denotes the set of all primes dividing  $n$ ; as usual,  $\pi(G) = \pi(|G|)$ , the set of all primes dividing the order of  $G$ .

A subgroup  $A$  of  $G$  is said to be  *$\mathfrak{U}$ -normal* in  $G$  [1] if either  $A \trianglelefteq G$  or  $A_G \neq A^G$  and every chief factor of  $G$  between  $A_G$  and  $A^G$  is cyclic.

Following L. Shemetkov [2], we use  $\sigma$  to denote some partition of  $\mathbb{P}$ . Thus  $\sigma = \{\sigma_i | i \in I\}$ , where  $\mathbb{P} = \bigcup_{i \in I} \sigma_i$  and  $\sigma_i \cap \sigma_j = \emptyset$  for all  $i \neq j$ . The symbol  $\sigma(n)$  denotes the set  $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ ;  $\sigma(G) = \sigma(|G|)$ .

The group  $G$  is said to be [3–5]:  *$\sigma$ -primary* if  $G$  is a  $\sigma_i$ -group for some  $i \in I$ ;  *$\sigma$ -nilpotent* if  $G = G_1 \times \dots \times G_n$  for some  $\sigma$ -primary groups  $G_1, \dots, G_n$ ;  *$\sigma$ -soluble* if every chief factor of  $G$  is  $\sigma$ -primary.

A set  $\mathcal{H}$  of subgroups of  $G$  is said to be a *complete Hall  $\sigma$ -set* of  $G$  [6; 7] if every member  $\neq 1$  of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $i \in I$  and  $\mathcal{H}$  contains exactly one Hall  $\sigma_i$ -subgroup of  $G$  for every  $i$  such that  $\sigma_i \cap \pi(G) \neq \emptyset$ .

A subgroup  $A$  of  $G$  is said to be [3]:  *$\sigma$ -permutable* in  $G$  if  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  such that  $AH^x = H^xA$  for all  $H \in \mathcal{H}$  and all  $x \in G$ ;  *$\sigma$ -subnormal* in  $G$  if there is a subgroup chain  $A = A_0 \leq A_1 \leq \dots \leq A_t = G$  such that either  $A_{i-1} \trianglelefteq A_i$  or  $A_i / (A_i - 1)_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, t$ .

Note that in the classical case when  $\sigma = \{\{2\}, \{3\}, \dots\}$ ,  $\sigma$ -permutable subgroups are also called  *$S$ -permutable* [8; 9], and in this case  $A$  is  $\sigma$ -subnormal in  $G$  if and only if it is subnormal in  $G$ .

The  $\sigma$ -permutable and  $\sigma$ -subnormal subgroups were studied by a lot of authors (see, in particular, the papers [3–6; 10–29]).

In this paper we consider some applications of the following generalisation of  $\sigma$ -subnormal and  $\sigma$ -permutable subgroups.



**Definition 1.** We say that a subgroup  $H$  of  $G$  is

(i) *partially  $\sigma$ -permutable* in  $G$  if there are a  $\mathfrak{U}$ -normal subgroup  $A$  and a  $\sigma$ -permutable subgroup  $B$  of  $G$  such that  $H = \langle A, B \rangle$ ;

(ii)  $(\mathfrak{U}, \sigma)$ -*embedded* in  $G$  if there are a partially  $\sigma$ -permutable subgroup  $S$  and a  $\sigma$ -subnormal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq S \leq H$ .

Note that every  $\mathfrak{U}$ -normal subgroup  $A = \langle A, 1 \rangle$  and every  $\sigma$ -permutable subgroup  $B = \langle 1, B \rangle$  are partially  $\sigma$ -permutable in  $G$ . Moreover, every partially  $\sigma$ -permutable subgroup  $S$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$  since in this case we have  $G = SG$  and  $S \cap G = S \leq S$ , where  $G$  is a  $\sigma$ -subnormal subgroup of  $G$  by definition.

Now we consider the following examples, which allow you to get various applications of the introduced concepts.

**Example 1.** (i) A subgroup  $H$  of  $G$  is said to be *weakly  $\sigma$ -permutable* [30] or *weakly  $\sigma$ -quasinormal* [31] in  $G$  if there is a  $\sigma$ -subnormal subgroup  $T$  and a  $\sigma$ -permutable subgroup  $S$  of  $G$  such that  $G = HT$  and  $H \cap T \leq S \leq H$ . Every weakly  $\sigma$ -quasinormal subgroup is  $(\mathfrak{U}, \sigma)$ -embedded in the group.

(ii) A subgroup  $H$  of  $G$  is said to be *weakly  $S$ -permutable* in  $G$  [32] if there are an  $S$ -permutable subgroup  $S$  and a subnormal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq S \leq H$ . It is clear that every weakly  $S$ -permutable subgroup is  $(\mathfrak{U}, \sigma)$ -embedded for every partition  $\sigma$  of  $\mathbb{P}$ .

(iii) Recall that a subgroup  $M$  of  $G$  is called *modular* in  $G$  if  $M$  is a modular element (in the sense of Kurosh [33, p. 43]) of the lattice  $\mathcal{L}(G)$  of all subgroups of  $G$ , that is (i)  $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$  for all  $X \leq G$ ,  $Z \leq G$  such that  $X \leq Z$ , and (ii)  $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$  for all  $Y \leq G$ ,  $Z \leq G$  such that  $M \leq Z$ .

A subgroup  $H$  of  $G$  is called  *$m$ - $\sigma$ -permutable* in  $G$  [34] if there are a modular subgroup  $A$  and a  $\sigma$ -permutable subgroup  $B$  of  $G$  such that  $H = \langle A, B \rangle$ . In view of [33, theorem 5.1.9], every modular subgroup is  $\mathfrak{U}$ -normal in the group. Therefore, every  $m$ - $\sigma$ -permutable subgroup is partially  $\sigma$ -permutable.

(iv) A subgroup  $H$  of  $G$  is called *weakly  $m$ - $\sigma$ -permutable* in  $G$  [34] if there are an  $m$ - $\sigma$ -permutable subgroup  $S$  and a  $\sigma$ -subnormal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T \leq S \leq H$ . It is clear that every weakly  $m$ - $\sigma$ -permutable subgroup is  $(\mathfrak{U}, \sigma)$ -embedded.

(v) A subgroup  $A$  of  $G$  is said to be  *$c$ -normal* in  $G$  [35] if for some normal subgroup  $T$  of  $G$  we have  $AT = G$  and  $A \cap T \leq A_G$ . Every  $c$ -normal subgroup is  $(\mathfrak{U}, \sigma)$ -embedded.

Our first observation generalises corresponding results in [34; 35].

**Theorem A.** (i) *If every non-nilpotent maximal subgroup of  $G$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ , then  $G$  is  $\sigma$ -soluble.*

(ii)  *$G$  is soluble if and only if every maximal subgroup of  $G$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$  and  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  whose members are soluble groups.*

In view of example 1 (iii), we get also from theorem A the following corollary.

**Corollary 1** [34, theorem B]. *If every non-nilpotent maximal subgroup of  $G$  is weakly  $m$ - $\sigma$ -permutable in  $G$ , then  $G$  is  $\sigma$ -soluble.*

In the case when  $\sigma = \{\{2\}, \{3\}, \dots\}$  we get from theorem A (ii) the following known result.

**Corollary 2** [35, theorem 3.1]. *If every maximal subgroup of  $G$  is  $c$ -normal in  $G$ , then  $G$  is soluble.*

Now, recall that if  $M_2 < M_1 < G$  where  $M_2$  is a maximal subgroup of  $M_1$  and  $M_1$  is a maximal subgroup of  $G$ , then  $M_2$  is said to be a *2-maximal subgroup* of  $G$ .

Our next theorem generalises a well-known Agrawal's result on supersolubility of groups with  $S$ -permutable 2-maximal subgroups.

**Theorem B.** *If every 2-maximal subgroup of  $G$  is partially  $\sigma$ -permutable in  $G$  and  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  whose members are supersoluble, then  $G$  is supersoluble.*

**Corollary 3.** *If every 2-maximal subgroup of  $G$  is  $\sigma$ -permutable in  $G$  and  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  whose members are supersoluble, then  $G$  is supersoluble.*

In the case when  $\sigma = \{\{2\}, \{3\}, \dots\}$  we get from theorem B the following known results.

**Corollary 4** [36; 37, chapter 1, theorem 6.5]. *If every 2-maximal subgroup of  $G$  is  $S$ -permutable in  $G$ , then  $G$  is supersoluble.*

**Corollary 5** [38]. *If every 2-maximal subgroup of  $G$  is modular in  $G$ , then  $G$  is supersoluble.*

Recall that  $G$  is *meta- $\sigma$ -nilpotent* [7] if  $G$  is an extension of a  $\sigma$ -nilpotent group by a  $\sigma$ -nilpotent group. An analysis of many open questions leads to the necessity of studying various classes of meta- $\sigma$ -nilpotent groups (see, for example, the recent papers [3; 11–18; 30] and the survey [7]).

Our next result gives the following characterisation of meta- $\sigma$ -nilpotent groups.

**Theorem C.** (i) *The following conditions are equivalent:*

(a)  *$G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  whose members are  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ ;*



(b)  $G$  is meta- $\sigma$ -nilpotent;

(c)  $G$  is  $\sigma$ -soluble and every  $\sigma$ -Hall subgroup  $H$  of  $G$  (that is  $\sigma(H) \cap \sigma(|G:H|) = \emptyset$ ) is  $c$ -normal in  $G$ .

(ii) If  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H}$  whose members are partially  $\sigma$ -permutable in  $G$ , then the derived subgroup  $G'$  of  $G$  is  $\sigma$ -nilpotent.

A group  $G$  is said to be: a  $D_\pi$ -group if  $G$  possesses a Hall  $\pi$ -subgroup  $E$  and every  $\pi$ -subgroup of  $G$  is contained in some conjugate of  $E$ ; a  $\sigma$ -full group of Sylow type [3] if every subgroup  $E$  of  $G$  is a  $D_{\sigma_i}$ -group for each  $\sigma_i \in \sigma(E)$ .

In view of example 1 (ii) we get from theorem C the following corollary.

**Corollary 6** [30, theorem 1.4]. *Let  $G$  be a  $\sigma$ -full group of Sylow type. If every Hall  $\sigma_i$ -subgroup of  $G$  is weakly  $\sigma$ -permutable in  $G$  for all  $\sigma_i \in \sigma(G)$ , then  $G$  is  $\sigma$ -soluble.*

In the case when  $\sigma = \{\{2\}, \{3\}, \dots\}$  we get from theorem C the following known result.

**Corollary 7** [39, chapter I, theorem 3.49].  *$G$  is metanilpotent if and only if every Sylow subgroup of  $G$  is  $c$ -normal.*

### Proof of theorem A

First we prove the following two lemmas.

**Lemma 1.** *Let  $A, B$  and  $N$  be subgroups of  $G$ , where  $A$  is partially  $\sigma$ -permutable in  $G$  and  $N$  is normal in  $G$ . Then:*

(1)  $AN/N$  is partially  $\sigma$ -permutable in  $G/N$ .

(2) If  $G$  is  $\sigma$ -full group of Sylow type and  $A \leq B$ , then  $A$  is partially  $\sigma$ -permutable in  $B$ .

(3) If  $G$  is  $\sigma$ -full group of Sylow type,  $N \leq B$  and  $B/N$  is partially  $\sigma$ -permutable in  $G/N$ , then  $B$  is partially  $\sigma$ -permutable in  $G$ .

(4) If  $G$  is  $\sigma$ -full group of Sylow type and  $B$  is partially  $\sigma$ -permutable in  $G$ , then  $\langle A, B \rangle$  is partially  $\sigma$ -permutable in  $G$ .

*Proof.* Let  $A = \langle L, T \rangle$ , where  $L$  is  $\mathfrak{U}$ -normal and  $T$  is  $\sigma$ -permutable subgroups of  $G$ .

(1)  $AN/N = \langle LN/N, TN/N \rangle$ , where  $LN/N$  is  $\mathfrak{U}$ -normal in  $G/N$  by [40, lemma 2.8 (2)] and  $TN/N$  is  $\sigma$ -permutable in  $G/N$  by [3, lemma 2.8 (2)]. Hence  $AN/N$  is partially  $\sigma$ -permutable in  $G/N$ .

(2) This follows from [3, lemma 2.8 (1); 40 lemma 2.8].

(3) Let  $B/N = \langle V/N, W/N \rangle$ , where  $V/N$  is  $\mathfrak{U}$ -normal in  $G/N$  and  $W/N$  is  $\sigma$ -permutable in  $G/N$ . Then  $B = \langle V, W \rangle$ , where  $V$  is  $\mathfrak{U}$ -normal in  $G$  by [40 lemma 2.8 (3)] and  $W$  is  $\sigma$ -permutable in  $G$ . Hence  $B$  is partially  $\sigma$ -permutable in  $G$ .

(4) Let  $B = \langle V, W \rangle$ , where  $V$  is  $\mathfrak{U}$ -normal and  $W$  is a  $\sigma$ -permutable subgroups of  $G$ . Then

$$\langle A, B \rangle = \langle \langle L, T \rangle, \langle V, W \rangle \rangle = \langle \langle L, V \rangle, \langle T, W \rangle \rangle,$$

where  $\langle L, V \rangle$  is  $\mathfrak{U}$ -normal in  $G$  by [40, lemma 2.8 (1)] and  $\langle T, W \rangle$  is  $\sigma$ -permutable in  $G$  by [3, lemma 2.8 (4)].

Hence  $\langle A, B \rangle$  is partially  $\sigma$ -permutable in  $G$ .

The lemma is proved.

**Lemma 2.** *Let  $A, B$  and  $N$  be subgroups of  $G$ , where  $A$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$  and  $N$  is normal in  $G$ .*

(1) *If either  $N \leq A$  or  $(|A|, |N|) = 1$ , then  $AN/N$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G/N$ .*

(2) *If  $G$  is  $\sigma$ -full group of Sylow type and  $A \leq B$ , then  $A$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $B$ .*

(3) *If  $G$  is  $\sigma$ -full group of Sylow type,  $N \leq B$  and  $B/N$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G/N$ , then  $B$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ .*

*Proof.* Let  $T$  be a  $\sigma$ -subnormal subgroup of  $G$  such that  $AT = G$  and  $A \cap T \leq S \leq A$  for some partially  $\sigma$ -permutable subgroup  $S$  of  $G$ .

(1) First note that  $NT \cap NA = (T \cap A)N$ . Therefore  $G/N = (AN/N)(TN/N)$  and

$$(AN/N) \cap (TN/N) = (AN \cap TN/N) = (A \cap T)N/N \leq SN/N,$$

where  $SN/N$  is a partially  $\sigma$ -permutable subgroup of  $G/N$  by lemma 1 (1). Hence  $AN/N$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G/N$ .

(2)  $B = A(B \cap T)$  and  $(B \cap T) \cap A = T \cap A \leq S \leq A$ , where  $S$  is partially  $\sigma$ -permutable in  $B$  by lemma 1 (2).

Hence  $A$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $B$ .

(3) See the proof of (1) and use lemma 1 (3).

The lemma is proved.

*Proof of theorem A.* (i) Assume that this assertion is false and let  $G$  be a counterexample of minimal order. Let  $R$  be a minimal normal subgroup of  $G$ .



(1)  $G/R$  is  $\sigma$ -soluble. Hence  $R$  is not  $\sigma$ -primary and it is a unique minimal normal subgroup of  $G$ .

Note that if  $M/R$  is a non-nilpotent maximal subgroup of  $G/R$ , then  $M$  is a non-nilpotent maximal subgroup of  $G$  and so it is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$  by hypothesis. Hence  $M/R$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G/R$  by lemma 2 (1). Therefore the hypothesis holds for  $G/R$ . Hence  $G/R$  is  $\sigma$ -soluble and so  $R$  is not  $\sigma$ -primary by the choice of  $G$ . Now assume that  $G$  has a minimal normal subgroup  $N \neq R$ . Then  $G/N$  is  $\sigma$ -soluble and  $N$  is not  $\sigma$ -primary. But, in view of the  $G$ -isomorphism  $RN/R \cong N$ , the  $\sigma$ -solubility of  $G/R$  implies that  $N$  is  $\sigma$ -primary. This contradiction completes the proof of (1).

In view of claim (1),  $R$  is not abelian. Hence  $|\pi(R)| > 1$ . Let  $p$  be any odd prime dividing  $|R|$  and  $R_p$  a Sylow  $p$ -subgroup of  $R$ .

(2) If  $G_p$  is a Sylow  $p$ -subgroup of  $G$  with  $R_p = G_p \cap R$ , then there is a maximal subgroup  $M$  of  $G$  such that  $RM = G$  and  $G_p \leq N_G(R_p) \leq M$ .

It is clear that  $G_p \leq N_G(R_p)$ . The Frattini argument implies that  $G = RN_G(R_p)$ . Conversely, claim (1) implies that  $N_G(R_p) \neq G$ , so for some maximal subgroup  $M$  of  $G$  we have  $RM = G$  and  $G_p \leq N_G(R_p) \leq M$ .

(3)  $M$  is not nilpotent and  $M_G = 1$ . Hence  $M$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ .

Assume that  $M$  is nilpotent, and let  $D = M \cap R$ . Then  $D$  is a normal subgroup of  $M$  and  $R_p$  is a Sylow  $p$ -subgroup of  $D$  since  $R_p \leq G_p \leq M$ . Hence  $R_p$  is characteristic in  $D$  and so it is normal in  $M$ . Therefore  $Z(J(R_p))$  is normal in  $M$ . Claims (1) and (2) imply that  $M_G = 1$ . Hence  $N_G(Z(J(R_p))) = M$  and so  $N_R(Z(J(R_p))) = D$  is nilpotent. This implies that  $R$  is  $p$ -nilpotent by the Glauberman – Thompson theorem on the normal  $p$ -complements. But then  $R$  is a  $p$ -group, contrary to claim (1). Hence we have (3).

(4) There is a  $\sigma$ -subnormal subgroup  $T$  of  $G$  such that  $MT = G$ ,  $M \cap T = 1$  and  $p$  does not divide  $|T|$ .

By claim (3), there are a partially  $\sigma$ -permutable subgroup  $S$  and a  $\sigma$ -subnormal subgroup  $T$  of  $G$  such that  $G = MT$  and  $M \cap T \leq S \leq M$ . Then  $S = \langle A, B \rangle$  for some  $\mathfrak{U}$ -normal subgroup  $A$  and  $\sigma$ -permutable subgroup  $B$  of  $G$ . Moreover, from the definition  $\mathfrak{U}$ -normality and claim (1) it follows that, in fact,  $S = B$  and  $S_G = 1$ . Suppose that  $S \neq 1$ . Then for every  $\sigma_i \in \sigma(S)$  we have  $S = O_{\sigma_i}(S) \times O_{\sigma_i}(S)$  by [3, theorem B]. Therefore for every Hall  $\sigma_i$ -subgroup  $H$  of  $G$  from  $SH = HS = O_{\sigma_i}(S)H$  we get that  $1 < O_{\sigma_i}(S) \leq H_G$ , contrary to claim (1). Therefore  $S = 1$ , so  $T \cap M = 1$ . Hence  $|T| = |G : M|$ , so  $p$  does not divide  $|T|$  since  $G_p \leq M$  by claim (2).

The final contradiction for (i). Let  $L$  be a minimal  $\sigma$ -subnormal subgroup of  $G$  contained in  $T$ . Then  $L$  is a simple group. If  $L$  is a  $\sigma_i$ -group for some  $i$ , then  $L \leq O_{\sigma_i}(G)$  by [12, lemma 2.2 (10)], which is impossible by claim (1).

Hence  $L$  is non-abelian, so it is subnormal in  $G$  by [12, lemma 2.2 (7)]. Suppose that  $L \not\leq R$ . Then  $L \cap R = 1$ . Conversely,  $R \leq N_G(L)$  by [41, chapter A, theorem 14.3]. Hence  $LR = L \times R$ , so  $L \leq C_G(R)$ . But claim (1) implies that  $R \not\leq C_G(R)$  and so  $C_G(R) = 1$ , a contradiction. Hence  $L$  is a minimal normal subgroup of  $R$ . It follows that  $p$  divides  $|L|$  and hence  $p$  divides  $|T|$ , contrary to claim (4). Therefore assertion (i) is true.

(ii) In view of theorem A, it is enough to show that if  $G$  is soluble, then every maximal subgroup  $M$  of  $G$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ . If  $M_G \neq 1$ , then  $M/M_G$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G/M_G$  by induction, so  $M$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$  by lemma 2 (3). Conversely, if  $M_G = 1$  and  $R$  is a minimal normal subgroup of  $G$ , then  $R$  is abelian and so  $G = R \rtimes M$ . Hence  $M$  is  $(\mathfrak{U}, \sigma)$ -embedded in  $G$ .

The theorem is proved.

### Proof of theorem B

**Lemma 3** [6, theorem A]. If  $G$  is  $\sigma$ -soluble, then  $G$  is a  $\sigma$ -full group of Sylow type.

**Lemma 4.** If  $G$  is  $\sigma$ -soluble and  $G$  possesses a complete Hall  $\sigma$ -set whose members are  $p$ -soluble, then  $G$  is  $p$ -soluble.

*Proof.* Suppose that this lemma is false and let  $G$  be a counterexample of minimal order. Let  $\mathcal{H} = \{H_1, \dots, H_t\}$  be a complete Hall  $\sigma$ -set of  $G$ . Then  $H_i$  is  $p$ -soluble by lemma 3 for all  $i$ .

First show that if  $R$  is minimal normal subgroup of  $G$ , then  $G/R$  is  $p$ -soluble. It is enough to show that the hypothesis holds for  $G/R$ .

Note that for every chief factor  $(H/R)/(K/R)$  of  $G/R$  we have that  $(H/R)/(K/R) \cong_G H/K$ , where  $H/K$  is a chief factor of  $G$  and  $H/K$  is  $\sigma$ -primary since  $G$  is  $\sigma$ -soluble. So  $(H/R)/(K/R)$  is  $\sigma$ -primary, hence  $G/R$  is  $\sigma$ -soluble.



Note also that  $\{H_1R/R, \dots, H_iR/R\}$  is a complete Hall  $\sigma$ -set of  $G/R$ , where  $H_iR/R \cong H_i/(H_i \cap R)$  is  $p$ -soluble since  $H_i$  is  $p$ -soluble. Therefore the hypothesis holds for  $G/R$ , so  $G/R$  is  $p$ -soluble by the choice of  $G$ .

Now show that  $R$  is  $p$ -soluble. Since  $G$  is  $\sigma$ -soluble,  $R$  is  $\sigma$ -primary, that is,  $\sigma_i$ -group for some  $i$ . Also, for every Hall  $\sigma_i$ -subgroup  $H$  of  $G$  we have  $R \leq H$ . So,  $R$  is  $p$ -soluble by the hypothesis, hence  $G$  is  $p$ -soluble.

The lemma is proved.

**Proof of theorem B.** Suppose that this theorem is false and let  $G$  be a counterexample of minimal order. Let  $\mathcal{H} = \{H_1, \dots, H_i\}$ . Then  $t > 1$  since  $H_1$  is supersoluble by hypothesis.

(1) *If  $R$  is minimal normal subgroup of  $G$ , then  $G/R$  is supersoluble. Hence  $R$  is the unique minimal normal subgroup of  $G$ ,  $R$  is not cyclic and  $R \not\leq \Phi(G)$ .*

It is enough to show that the hypothesis holds for  $G/R$ . First note that  $\{H_1R/R, \dots, H_iR/R\}$  is a complete Hall  $\sigma$ -set of  $G/R$ , where  $H_iR/R \cong H_i/(H_i \cap R)$  is supersoluble since  $H_i$  is supersoluble by hypothesis.

Now assume that statement (1) is false. Then  $G/R$  is not nilpotent, so every Sylow  $p$ -subgroup in  $G/R$  is proper. Then for every Sylow  $p$ -subgroup  $P$  of  $G/R$  it follows that  $P$  is contained in some maximal subgroup of  $G/R$ . Hence  $R$  is contained in some 2-maximal subgroup  $T$  of  $G$  and so  $T$  is partially  $\sigma$ -permutable in  $G$  by the hypothesis. But then  $T/R$  is 2-maximal of  $G/R$  and partially  $\sigma$ -permutable in  $G/R$  by lemma 1 (1). Therefore the hypothesis holds for  $G/R$ , so  $G/R$  is supersoluble by the choice of  $G$ . Then we have a contradiction.

Moreover, it is well-known that the class of all supersoluble groups is a saturated formation [42 chapter VI, definition 8.6]. Hence the choice of  $G$  implies that  $R$  is the unique minimal normal subgroup of  $G$ ,  $R$  is not cyclic and  $R \not\leq \Phi(G)$ . Hence we have (1).

(2)  *$G$  is soluble.*

Every 2-maximal subgroup in  $G$  is partially  $\sigma$ -permutable and so, partially  $\sigma$ -subnormal in  $G$  by [3, theorem B]. Then, in view of theorem in [40],  $G$  is  $\sigma$ -soluble. Hence, from lemma 4 it follows that  $G$  is soluble. So we have (2).

(3)  *$R = O_p(G) \not\leq \Phi(G)$  for some prime  $p \in \sigma_i$ . Hence for some maximal subgroup  $M$  of  $G$  we have  $G = R \rtimes M$  and  $M \neq M_G = 1$ .*

By claim (2),  $G$  is soluble and so  $R$  is a  $p$ -group for some  $p \in \sigma_i$ . Hence the choice of  $G$  and claim (1) imply that  $R$  is a unique minimal normal subgroup of  $G$ . Moreover,  $R \not\leq \Phi(G)$  by claim (1), so  $R = C_G(R) = O_p(G)$  by [41, chapter A, lemma 15.2]. Hence for some maximal subgroup  $M$  of  $G$  we have  $G = R \rtimes M$  and  $M \neq M_G = 1$  by claim (1).

(4) *If  $1 < H \leq M$ , then  $H$  is not  $\mathcal{U}$ -normal in  $G$ .*

Indeed, if  $H$  is  $\mathcal{U}$ -normal in  $G$ , then  $H^G/H_G \leq Z_{\mathcal{U}}(G/H_G)$ , where  $H_G = 1$  by claim (3). Hence  $R \leq H^G \leq Z_{\mathcal{U}}(G)$  by claim (1). But then  $R$  is cyclic, contrary to claim (1). This contradiction completes the proof of the claim.

(5)  *$M$  is not a group of prime order.*

Suppose that  $|M| = q$  for some prime  $q$ . Hence  $|M| = |G : R|$  is a prime and so  $R$  is a maximal subgroup of  $G$ . Then every maximal subgroup  $V$  of  $R$  is 2-maximal in  $G$ , so  $V$  is partially  $\sigma$ -permutable in  $G$  by hypothesis. So  $V = \langle A, B \rangle$ , where  $A$  is  $\mathcal{U}$ -normal and  $B$  is  $\sigma$ -permutable in  $G$ . Assume  $A \neq 1$ . Note  $A_G = 1$  by the minimality of  $R$ . Then  $R \leq A^G \leq Z_{\mathcal{U}}(G)$  and so  $R$  is cyclic, contrary to claim (1). Hence  $V = B$  is  $\sigma$ -permutable in  $G$ . Therefore every maximal subgroups of  $R$  is  $\sigma$ -permutable in  $G$ .

Note that  $R \leq H_i$  since  $R$  is  $\sigma_i$ -group by claim (3) and  $H_i = R \rtimes (H_i \cap M)$ , again by claim (3). Since  $H_i$  is supersoluble by hypothesis, some maximal subgroup  $W$  of  $R$  is normal in  $H_i$ . In addition,  $W$  is  $\sigma$ -permutable in  $G$  since it is a maximal subgroup of  $R$ . Hence for each  $j \neq i$  we have  $WH_j = H_jW$ , which implies that  $H_j \leq N_G(W)$  since  $R \cap WH_j = W(R \cap H_j) = W$ . Therefore  $W$  is normal in  $G$ , so the minimality of  $R$  implies that  $W = 1$  and hence  $|R| = p$ , which is impossible by claim (3). Hence we have (5).

(6) *If  $T$  is a maximal subgroup of  $M$ , then  $T^G$  is a  $\sigma_i$ -subgroup of  $G$ .*

Indeed,  $T$  is partially  $\sigma$ -permutable in  $G$  by hypothesis, so  $T = \langle A, B \rangle$  for some  $\mathcal{U}$ -normal subgroup  $A$  and some  $\sigma$ -permutable subgroup  $B$  of  $G$ . Note that  $T \neq 1$  by claim (5). Conversely,  $A = 1$  by claim (4) and so  $T = B$  is  $\sigma$ -permutable in  $G$ . Therefore  $T^G/T_G$  is  $\sigma$ -nilpotent [3, theorem B (ii)]. We have  $T_G \leq M_G = 1$ , so  $T^G/T_G \cong T^G/1 \cong T^G$  is  $\sigma$ -nilpotent group. Hence the subgroup  $O_{\sigma_k}(T^G)$  is characteristic in  $T^G$ , so it is normal in  $G$ . By claim (3) we have that  $O_{\sigma_k}(T^G) = 1$  for all  $k \neq i$ . Hence  $T^G = O_{\sigma_i}(T^G)$  is a  $\sigma_i$ -subgroup of  $G$ .

(7)  *$M$  is not  $\sigma_i$ -group.*

Suppose that this is false and let  $T$  be a maximal subgroup of  $M$ . Then  $T \neq 1$  by claim (5). Conversely,  $T$  is a  $\sigma_i$ -group by the hypothesis and  $T^G$  is a  $\sigma_i$ -subgroup of  $G$  by claim (6). Then we have a contradiction. Hence,  $M$  is not a  $\sigma_i$ -group.



(8)  $M$  is not  $\sigma_i$ -group (this follows from the facts that  $t > 1$  and  $R$  is a  $\sigma_i$ -group).

*Final contradiction.*

Let  $T$  be a maximal subgroup of  $M$ , containing a Hall  $\sigma_i$ -subgroup of  $M$ . Then  $T^G$  is  $\sigma_i$ -group by claim (6). Therefore, a Hall  $\sigma_i$ -subgroup of  $M$  is the identity group. Hence  $M$  is  $\sigma_i$ -group, contrary to claim (8). This contradiction completes the proof of the result.

### Proof of theorem C

We use  $\mathfrak{R}_\sigma$  to denote the class of all  $\sigma$ -nilpotent groups.

**Lemma 5** [3, corollary 2.4 and lemma 2.5]. (1) *The class  $\mathfrak{R}_\sigma$  is closed under taking products of normal subgroups, homomorphic images and subgroups.*

(2) *If  $G/N$  and  $G/R$  are  $\sigma$ -nilpotent, then  $G/(N \cap R)$  is  $\sigma$ -nilpotent.*

(3) *If  $E$  is a normal subgroup of  $G$  and  $E/(E \cap \Phi(G))$  is  $\sigma$ -nilpotent, then  $E$  is  $\sigma$ -nilpotent.*

Recall that  $G^{\mathfrak{R}_\sigma}$  denotes the  $\sigma$ -nilpotent residual of  $G$ , that is, the intersection of all normal subgroups  $N$  of  $G$  with  $\sigma$ -nilpotent quotient  $G/N$ . In view of [43, proposition 2.2.8], we get from lemma 5 (1) the following result.

**Lemma 6.** *If  $N$  is a normal subgroup of  $G$ , then  $(G/N)^{\mathfrak{R}_\sigma} = G^{\mathfrak{R}_\sigma}N/N$ .*

The next lemma is proved by the direct verifications on the basis of lemmas 5 and 6.

**Lemma 7.** (1)  *$G$  is meta- $\sigma$ -nilpotent if and only if  $G^{\mathfrak{R}_\sigma}$  is  $\sigma$ -nilpotent.*

(2) *If  $G$  is meta- $\sigma$ -nilpotent, then every quotient  $G/N$  of  $G$  is meta- $\sigma$ -nilpotent.*

(3) *If  $G/N$  and  $G/R$  are meta- $\sigma$ -nilpotent, then  $G/(N \cap R)$  is meta- $\sigma$ -nilpotent.*

(4) *If  $E$  is a normal subgroup of  $G$  and  $E/(E \cap \Phi(G))$  is meta- $\sigma$ -nilpotent, then  $E$  is meta- $\sigma$ -nilpotent.*

**Lemma 8.** *Let  $A$ ,  $B$  and  $N$  be subgroups of  $G$ , where  $A$  is  $c$ -normal in  $G$  and  $N$  is normal in  $G$ .*

(1) *If either  $N \leq A$  or  $(|A|, |N|) = 1$ , then  $AN/N$  is  $c$ -normal in  $G/N$ .*

(2) *If  $N \leq B$  and  $B/N$  is  $c$ -normal in  $G/N$ , then  $B$  is  $c$ -normal in  $G$ .*

*Proof.* See the proof of lemma 2.

A natural number  $n$  is said to be a  $\Pi$ -number if  $\sigma(n) \subseteq \Pi$ . A subgroup  $A$  of  $G$  is said to be: a *Hall  $\Pi$ -subgroup* of  $G$  [6; 7] if  $|A|$  is a  $\Pi$ -number and  $|G:A|$  is a  $\Pi'$ -number; a  *$\sigma$ -Hall subgroup* of  $G$  if  $A$  is a Hall  $\Pi$ -subgroup of  $G$  for some  $\Pi \subseteq \sigma$ .

Recall also that a normal subgroup  $E$  of  $G$  is called *hypercyclically embedded* in  $G$  [33, p. 217] if every chief factor of  $G$  below  $E$  is cyclic.

*Proof of theorem C.* Let  $D = G^{\mathfrak{R}_\sigma}$  be the  $\sigma$ -nilpotent residual of  $G$ .

(i) (a)  $\Rightarrow$  (b). Assume that this is false and let  $G$  be a counterexample of minimal order. Then  $D$  is not  $\sigma$ -nilpotent since  $G/D$  is  $\sigma$ -nilpotent by lemma 5 (2). Let  $\mathcal{H} = \{H_1, \dots, H_t\}$ . We can assume without loss of generality that  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ . Let  $S_i$  be a partially  $\sigma$ -permutable subgroup and  $T_i$  be a  $\sigma$ -subnormal subgroup of  $G$  such that  $S_i \leq H_i$ ,  $H_i T_i = G$  and  $H_i \cap T_i \leq S_i$  for all  $i = 1, \dots, t$ . Then, for every  $i$ ,  $S_i = \langle A_i, B_i \rangle$  for some  $\mathfrak{U}$ -normal subgroup  $A_i$  and  $\sigma$ -permutable subgroup  $B_i$  of  $G$ .

(1) *If  $R$  is a  $\sigma$ -primary minimal normal subgroup of  $G$ , then  $G/R$  is meta- $\sigma$ -nilpotent and so  $G$  is  $\sigma$ -soluble. Moreover,  $R$  is a unique minimal normal subgroup of  $G$ ,  $C_G(R) \leq R$  and  $R$  is not cyclic.*

First we show that  $G/R$  is meta- $\sigma$ -nilpotent. In view of the choice of  $G$ , it is enough to show that the hypothesis holds for  $G/R$ . Since  $R$  is  $\sigma$ -primary, for some  $i$  we have  $R \leq H_i$  and  $(|R|, |H_j|) = 1$  for all  $j \neq i$ . Therefore  $\{H_1 R/R, \dots, H_t R/R\}$  is a complete Hall  $\sigma$ -set of  $G/R$  whose members are  $(\mathfrak{U}, \sigma)$ -embedded in  $G/R$  by lemma 2 (1). Hence the hypothesis holds for  $G/R$ , so  $G/R$  is meta- $\sigma$ -nilpotent and  $G$  is  $\sigma$ -soluble. Hence every minimal normal subgroup of  $G$  is  $\sigma$ -primary, so  $R$  is a unique minimal normal subgroup of  $G$  and  $R \not\leq \Phi(G)$  by lemma 7 (4). Hence  $C_G(R) \leq R$  by [41, chapter A, lemma 15.6]. Finally, note that in the case when  $R$  is cyclic we have  $|R| = p$  for some prime  $p$  and so  $G/C_G(R) = G/R$  is cyclic, which implies that  $G$  is metanilpotent and so it is meta- $\sigma$ -nilpotent. Therefore we have (1).

(2) *For some  $i$ ,  $i = 1$  say, we have  $S_i = S_1 \neq 1$ . Moreover, if for some  $k$  we have  $S_k = 1$ , then  $T_k$  is a normal complement to  $H_k$  in  $G$ .*

Assume that  $S_i = 1$ . Then  $H_i \cap T_i = 1$ , so  $T_i$  is a  $\sigma$ -subnormal Hall  $\sigma'_i$ -subgroup of  $G$ . Hence  $T_i$  is a normal complement to  $H_i$  in  $G$  by [3, lemma 2.6 (10)]. Moreover,  $G/T_i \simeq H_i$  is  $\sigma$ -nilpotent. Suppose that  $S_i = 1$  for all  $i = 1, \dots, t$ . Then  $T_1 \cap \dots \cap T_t = 1$  by [41, chapter A, theorem 1.6 (b)], so

$$G \simeq G/1 = G/(T_1 \cap \dots \cap T_t)$$

is  $\sigma$ -nilpotent by lemma 5 (2). Then we have a contradiction. Hence for some  $i$  we have  $S_i \neq 1$ .



(3) If  $S_i \neq 1$ , then  $(H_i)_G \neq 1$ .

Assume that this is false. Then every non-identity subgroup  $L$  of  $H_i$  is not  $\sigma$ -permutable in  $G$  since otherwise for every  $x \in G$  we have  $LH_i^x = H_i^xL = H_i^x$  which implies that  $1 < L \leq (H_i)_G = 1$ .

Therefore  $B_i = 1$  and so  $S_i = A_i$  is a  $\mathfrak{U}$ -normal subgroup of  $G$  with  $(S_i)_G = 1$ . But then we have  $1 < (S_i)^G$  is hypercyclically embedded in  $G$  by the definition  $\mathfrak{U}$ -normality and so  $R$  is cyclic, contrary to claim (1). Hence we have (3).

(4)  $G$  possesses a  $\sigma$ -primary minimal normal subgroup,  $R$  say.

Claims (2) and (3) imply that  $(H_1)_G \neq 1$ . Therefore, if  $R$  is a minimal normal subgroup of  $G$  contained in  $(H_1)_G$ , then  $R$  is  $\sigma$ -primary.

The final contradiction for the implication (a)  $\Rightarrow$  (b). Claims (1) and (4) imply that  $G$  is  $\sigma$ -soluble, so  $R$  is a unique minimal normal subgroup of  $G$  by claim (1). Hence claims (2) and (3) imply that  $T_2, \dots, T_t$  are normal subgroups of  $G$  and  $G/T_k \simeq H_k$  for all  $k = 2, \dots, t$ . Hence  $G/(T_2 \cap \dots \cap T_t)$  is  $\sigma$ -nilpotent by lemma 5 (2). Conversely,  $T_2 \cap \dots \cap T_t = H_1$  by [41, chapter A, theorem 1.6 (b)] and so  $G$  is meta- $\sigma$ -nilpotent, contrary to the choice of  $G$ . This contradiction completes the proof of the (a)  $\Rightarrow$  (b).

(b)  $\Rightarrow$  (c). The subgroup  $D$  is  $\sigma$ -nilpotent by lemma 7 (1). Let  $\Pi = \sigma(H)$ . Then  $H$  is a Hall  $\Pi$ -subgroup of  $G$ .

Suppose that  $H_G \neq 1$ . Then  $H/H_G$  is  $c$ -normal in  $G$  by induction since the hypothesis holds for  $G/H_G$  by lemma 7 (2). Hence  $H$  is  $c$ -normal in  $G$  by lemma 8 (2).

Now assume that  $H_G = 1$ . Then, since  $D$  is  $\sigma$ -nilpotent, it follows that  $D \cap H = 1$ . Conversely,  $G/D$  is  $\sigma$ -nilpotent by lemma 5 (2) and  $H \simeq HD/D$  is a Hall  $\Pi$ -subgroup of  $G/D$ , so  $HD/D$  has a normal complement  $T/D$  in  $G/D$ . Then  $T$  is a normal subgroup of  $G$  such that  $HT = G$  and  $T \cap H \leq T \cap HD \cap H \leq D \cap H = 1$ . Hence  $H$  is  $c$ -normal in  $G$ . Therefore the implication (b)  $\Rightarrow$  (c) holds.

(c)  $\Rightarrow$  (b). In view of example 1 (v), this application is a corollary of the implication (a)  $\Rightarrow$  (b).

(ii) Suppose that this assertion is false and let  $G$  be a counterexample of minimal order. Then  $G$  is not  $\sigma$ -nilpotent, so  $|\sigma(G)| > 1$ . Moreover, from part (i) we know that  $D$  is  $\sigma$ -nilpotent and so  $G$  is  $\sigma$ -soluble. Let  $\mathcal{H} = \{H_1, \dots, H_t\}$ .

Let  $R$  be a minimal normal subgroup of  $G$ . Then  $R$  is a  $\sigma_i$ -group for some  $i$ , so the hypothesis holds for  $G/R$  by lemma 1 (1). Hence  $(G/R)' = G'R/R = G'/(G' \cap R)$  is  $\sigma$ -nilpotent by the choice of  $G$ . Therefore  $R \leq G'$  and  $R \not\leq \Phi(G)$  by lemma 5 (3). Moreover, if  $G$  has a minimal normal subgroup  $N \neq R$  of  $G$ , then  $N \leq G'$  and  $G' \simeq G'/1 = G'/(R \cap N)$  is  $\sigma$ -nilpotent, contrary to the choice of  $G$ . Therefore  $R$  is a unique minimal normal subgroup of  $G$  and  $C_G(R) \leq R$  by [41, chapter A, lemma 16.5]. We can assume without loss of generality that  $R \leq H_1$ .

Let  $M$  be a maximal subgroup of  $G$  such that  $R \not\leq M$ . Then  $M_G = 1$  and  $|G : M|$  is a  $\sigma_i$ -number. Therefore for some  $x \in G$  we have  $H = H_2^x \leq M$ . Then  $H = \langle A, B \rangle$  for some  $\mathfrak{U}$ -normal subgroup  $A$  and  $\sigma$ -permutable subgroup  $B$  of  $G$ . Moreover,  $A_G \leq M_G = 1$ , so  $A^G$  is hypercyclically embedded in  $G$  by the definition  $\mathfrak{U}$ -normality. If  $A \neq 1$ , then  $R \leq A^G$  and so  $|R| = p$  for some prime  $p$ . But then  $C_G(R) = R$  and  $G/R = G/C_G(R)$  is cyclic. Hence  $G'$  is nilpotent. This contradiction shows that  $A = 1$ , so  $H = B$  is  $\sigma$ -permutable in  $G$ . But then  $HH^x = H^xH = H$  for all  $x \in G$  since  $H$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $i$ . Hence  $H$  is normal in  $G$ , so  $1 < H \leq M_G$ , a contradiction. Therefore assertion (ii) is true.

The theorem is proved.

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