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ОБ АППРОКСИМАЦИЯХ СОПРЯЖЕННЫХ ФУНКЦИЙ И ИХ ПРОИЗВОДНЫХ НА ОТРЕЗКЕ ЧАСТИЧНЫМИ СУММАМИ РЯДОВ ФУРЬЕ – ЧЕБЫШЕВА

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Аннотация. Изучены аппроксимации сопряженных функций на отрезке $[-1, 1]$ с плотностью $f \in H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, сопряженными рядами Фурье – Чебышева. Установлены порядковые оценки приближений, зависящие от положения точки на отрезке. Отмечено, что приближения на концах отрезка имеют бóльшую скорость убывания

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в сравнении со всем отрезком. Введены классы функций, которые можно в некотором смысле ассоциировать с производной сопряженной функции на отрезке $[-1, 1]$, и изучены приближения функций из этих классов частичными суммами рядов Фурье – Чебышева. Найдено интегральное представление приближений. При плотности $f \in W^1 H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, устанавливаются порядковые оценки приближений, также зависящие от положения точки на отрезке. Рассмотрен случай, когда плотность $f(t) = |t|^s$, $s > 1$. При этом получены интегральное представление приближений, оценки поточечных и равномерных приближений, асимптотическая оценка равномерных приближений. Отмечено, что порядки равномерных приближений изучаемой функции частичными суммами ряда Фурье – Чебышева и соответствующей ей сопряженной функции сопряженными суммами совпадают.

Ключевые слова: сингулярный интеграл на отрезке; сопряженная функция; условие Липшица; ряд Фурье – Чебышева; равномерные оценки; асимптотические оценки.

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ON THE APPROXIMATION OF CONJUGATE FUNCTIONS AND THEIR DERIVATIVES ON THE SEGMENT BY PARTIAL SUMS OF FOURIER – CHEBYSHEV SERIES

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Abstract. In this paper, we study the approximation of conjugate functions with the density $f \in H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, on the segment $[-1, 1]$ by the conjugate Fourier – Chebyshev series. We establish the order estimations of the approximation depending on the location of a point on the segment. It is noted that approximation at the endpoints of the segment has a higher rate of decrease in comparison with the whole segment. We introduce classes of functions, which, in a certain sense, can be associated with the derivative of a conjugate function on the segment $[-1, 1]$, and the approximation of functions from these classes by partial sums of the Fourier – Chebyshev series is studied. An integral representation of the approximation is found. In the case when the density $f \in W^1 H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, the order estimations of the approximation, depending on the location of the point on the segment, are established. The case, when the density $f(t) = |t|^s$, $s > 1$, is considered. In this case, an integral representation of the approximation, estimations for pointwise and uniform approximations, as well as an asymptotic estimation for the uniform approximation are obtained. It is noted that the order of the uniform approximations of the function under study by partial sums of the Fourier – Chebyshev series and the corresponding conjugate function by conjugate sums coincide.

Keywords: singular integral on a segment; conjugate function; Lipschitz condition; Fourier – Chebyshev series; uniform estimations; asymptotic estimations.

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Introduction

The integral (in the sense of the Cauchy principal value) with a Cauchy-type kernel

$$\hat{f}(x) = \frac{\sqrt{1 - x^2}}{\pi} \int_{-1}^{+1} \frac{f(t)}{t - x} \frac{dt}{\sqrt{1 - t^2}}, \ x \in [-1, 1], \tag{1}$$

plays an important role in various fields of mathematics and physics (see, for example, [1; 2]). For its existence it suffices to assume that the density $f(t)$ satisfies the Lipschitz condition of any order on the segment $[-1, 1]$. It is known (see [3]) that the transformation $\hat{f}$ can be considered as one of the ways for defying the conjugate function on the segment $[-1, 1]$. We also associate $\hat{f}$ with the conjugate Fourier – Chebyshev series (this series converges under the above said constrains):

$$\hat{f}(x) = \sum_{n=1}^{+\infty} c_n \sin(n \arccos x), \tag{2}$$

where

$$c_n = \frac{2}{\pi}\int_{-1}^{+1} f(t) T_n(t) \frac{dt}{\sqrt{1-t^2}},\ T_n(t) = \cos(n \arccos t),\ n = 1, 2, \ldots,$$

are the Fourier – Chebyshev coefficients. Note that $\hat{f}(\cos\theta)$ can be expressed in terms of the conjugate function to $f(\cos\theta)$ using the singular integral with a Hilbert kernel

$$\hat{f}(\cos\theta) = -\frac{1}{2\pi}\int_{0}^{2\pi} f(\cos\tau)\operatorname{ctg}\frac{\tau-\theta}{2}d\tau,\ \theta \in [0, \pi].$$

Also it can be associated with the corresponding conjugate trigonometric Fourier series.

The study of conjugate functions in the trigonometric case began with the works of J. Priwaloff [4; 5], A. Kolmogoroff [6], M. Riesz [7; 8]. Here we put special emphasis on the following result. Let $\bar{H}^{(\alpha)}$, $0 \le \alpha \le 1$, be a class of conjugate functions with a density that satisfies the Lipschitz condition of order α. The exact upper bounds of the deviations of partial sums of the conjugate 2π-periodic Fourier series from the functions of classes $\bar{H}^{(\alpha)}$ were found by S. Nikol'skii [9].

V. Motorny (see, for example, [10; 11]) studied approximations of singular integrals of the form (1) with a density belonging to certain classes of continuous functions on the interval $[-1, 1]$. V. Misiuk and A. Pekarskii [12] solved the classical problem of N. Bari [13] and S. Stechkin [14] about the best approximation of functions and their conjugates on a segment by algebraic polynomials.

The method of approximation of continuous functions on the segment $[-1, 1]$, based on the Fourier – Chebyshev series, has wide applications. Here we should mention the works of S. Nikol'skii [15], A. Timan [16], I. Ganzburg [17], Yu. Rusetskii [18], I. Ganzburg and A. Timan [19]. At the same time, the study of the approximation properties of the conjugate Fourier – Chebyshev series (2) was episodic. For example, pointwise and uniform approximations of the singular integral with a Hilbert kernel with a density having a power singularity by partial sums of the conjugate Fourier – Chebyshev series were studied in [20].

In the first part of this work, we study the approximation of the conjugate functions of form (1) with the density $f \in H^{(\alpha)}[-1, 1]$, $0 < \alpha \le 1$, by partial sums of the conjugate Fourier – Chebyshev series

$$\hat{s}_n(f, x) = \sum_{k=1}^{n} c_k \sin(k \arccos x),\ c_k = \frac{2}{\pi}\int_{-1}^{+1} f(t) T_k(t) \frac{dt}{\sqrt{1-t^2}},\ x \in [-1, 1].$$

Let the series

$$\sum_{k=1}^{+\infty} k c_k \cos(k \arccos x),\ x \in [-1, 1],$$

be the Fourier – Chebyshev series of a summable on the segment $[-1, 1]$ function. We consider the class of functions that can be represented as follows:

$$\tilde{f}(x) = -\sum_{k=1}^{+\infty} k c_k \cos(k \arccos x),\ x \in [-1, 1]. \tag{3}$$

Interest in the study of such functions is due to their relationship with conjugate series (2). Indeed, in this case (2) and (3) clearly imply the relation

$$\tilde{f}(x) = \sqrt{1-x^2}\,\hat{f}'(x),\ x \in [-1, 1].$$

Functions of form (3) have, in a certain sense, a periodic analogue. Let r and β be fixed real numbers ($r > 0$) and the series

$$\sum_{k=1}^{+\infty} k^r \left[a_k \cos\left(kt + \frac{\pi\beta}{2}\right) + b_k \sin\left(kt + \frac{\pi\beta}{2}\right)\right], \tag{4}$$

be a Fourier series of some summable function. Then this function is called (r, β)-derivative of the function f in the Weyl – Nagy sense, it is denoted by $f_\beta^r(\cdot)$ (see, for example, [21; 22]), and in addition, a_k, b_k, $k \in \mathbb{N}$, are Fourier coefficients of the function f. The set of functions f that satisfy such a condition is denoted by W_β^r. If, in addition, $f_\beta^r \in H^{(\alpha)}$, $0 < \alpha \le 1$, that is, it satisfies the Lipschitz condition of order α, then we say that f belongs to the class $W_\beta^r H^{(\alpha)}$. The classes W_β^r, $r > 0$, were introduced by S. Stechkin [23]. Approximation problems on them were the subject of research by many specialists in the theory of functions [24]. For example, the approximation properties of various summation methods of the trigonometric Fourier series on classes W_β^r were studied by the representatives of the Ukrainian mathematical school (see [25; 26]).

The conjugate function (2) can be written as follows:

$$\hat{f}(x) = \sum_{k=1}^{+\infty} c_k \cos\left(k \arccos x - \frac{\pi}{2}\right).$$

Taking into account the expression (4) we say, that functions of form (3) are algebraic analogue of its $(1, -1)$-derivatives in the Weyl – Nagy sense on the segment $[-1, 1]$.

In the second part of the paper, approximations of functions from class (3) by partial sums of the Fourier – Chebyshev series

$$s_n(\tilde{f}, x) = \frac{\tilde{c}_0}{2} + \sum_{k=1}^{n} \tilde{c}_k T_k(x), \; \tilde{c}_k = \frac{2}{\pi}\int_{-1}^{+1} \tilde{f}(t) T_k(t) \frac{dt}{\sqrt{1-t^2}}, \; k = 0, 1, \ldots, n,$$

are considered. Integral representations of the approximation are established. Also, we obtain estimations of the approximation when the density $f \in W^1 H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$. It should be mentioned that here we use the methods proposed by O. Besov (see, for example, [27; 28]). The obtained estimations depend on the location of the point on the segment $[-1, 1]$. Moreover, it appears that approximation at the endpoints of the segment is better in order than on the entire segment.

In the third part of the work, approximations of individual functions belonging to functional class (3), when the density $f(t) = |t|^s$, $s > 1$, are investigated. For the study of the asymptotic behaviour of integrals the Laplace method [29; 30] is used.

Approximation of conjugate functions with a density satisfying Lipschitz condition

Let

$$\hat{\varepsilon}_n(f, x) = \hat{f}(x) - \hat{s}_n(f, x), \; x \in [-1, 1], \tag{5}$$

$$\hat{\varepsilon}_n(f) = \left\| \hat{f}(x) - \hat{s}_n(f, x) \right\|_{C[-1, 1]}, \; n \in \mathbb{N}, \tag{6}$$

where $\hat{s}_n(f, x)$ are partial sums of its conjugate Fourier – Chebyshev series defined in (2).

Theorem 1. *For approximation of conjugate function (1) with the density* $f \in H^{(\alpha)}[-1, 1]$, $0 < \alpha \le 1$, *on the segment* $[-1, 1]$ *by partial sums of its conjugate Fourier – Chebyshev series the following estimation holds for sufficiently large n:*

$$\left|\hat{\varepsilon}_n(f, x)\right| \le \begin{cases} 2\pi^\alpha \left(\sqrt{1-x^2}\right)^\alpha \dfrac{\ln n}{n^\alpha} + \dfrac{\pi^{2\alpha}|x|^\alpha}{2\alpha n^\alpha} + \dfrac{\pi^{2\alpha} \ln n}{2^{1+\alpha} n^{2\alpha}}, \; \alpha \in (0, 1), \\ \pi^2 \sqrt{1-x^2} \dfrac{\ln n}{n} + \dfrac{\pi^2 |x|}{2n} + \pi^2 \dfrac{\ln n}{4n^2}, \; \alpha = 1. \end{cases} \tag{7}$$

P r o o f. Let us consider the deviation

$$\hat{\varepsilon}_n(f, x) = \hat{f}(x) - \hat{s}_n(f, x) = \sum_{k=n+1}^{+\infty} c_k \sin(k \arccos x), \; x \in [-1, 1],$$

of partial sums of conjugate series (2) from conjugate function (1). It is well-known (see [20]) that for the series remainder the following integral representation holds:

$$\hat{\varepsilon}_n(f, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\cos\tau) \hat{D}_n(\tau - \theta) d\tau, \; x = \cos\theta, \; x \in [-1, 1],$$

where

$$\hat{D}_n(\tau) = \frac{\cos\left(n + \frac{1}{2}\right)\tau}{\sin\frac{\tau}{2}}. \tag{8}$$

It is not difficult to see that

$$\hat{\varepsilon}_n(f, x) = \frac{1}{2\pi}\int_0^{\pi}\left[f\left(\cos(\theta + \tau)\right) - f\left(\cos(\theta - \tau)\right)\right]\hat{D}_n(\tau)d\tau,\ x = \cos\theta,\ x \in [-1, 1]. \tag{9}$$

Here the integrand is even, 2π-periodic with respect to the integration variable. Therefore

$$\hat{\varepsilon}_n(f, x) = \frac{1}{4\pi}\int_0^{2\pi}\psi_\theta(\tau)\cos\lambda_1\tau d\tau,\ \lambda_1 = n + \frac{1}{2},$$

where

$$\psi_\theta(\tau) = \frac{f\left(\cos(\theta + \tau)\right) - f\left(\cos(\theta - \tau)\right)}{\sin\frac{\tau}{2}}.$$

For the further proof we use the methods proposed by O. Besov (see, for example, [27; 28]). Taking into account the properties of the integrand, the latter representation can be written as

$$\hat{\varepsilon}_n(f, x) = \frac{1}{8\pi}\int_0^{2\pi}\left[\psi_\theta(\tau) - \psi_\theta\left(\tau + \frac{\pi}{\lambda_1}\right)\right]\cos\lambda_1\tau d\tau,\ \lambda_1(n) = n + \frac{1}{2}.$$

Further for brevity we will write simply λ_1, assuming that λ_1 depends on n.

We split the integral in the right-hand side into three integrals over the segments $\left[0, \frac{\pi}{\lambda_1}\right]$, $\left[\frac{\pi}{\lambda_1}, 2\pi - \frac{\pi}{\lambda_1}\right]$ and $\left[2\pi - \frac{\pi}{\lambda_1}, 2\pi\right]$, so that

$$\hat{\varepsilon}_n(f, x) = \frac{1}{8\pi}\left[I_1 + I_2 + I_3\right],\ x = \cos\theta,\ x \in [-1, 1], \tag{10}$$

where

$$I_1 = \int_0^{\frac{\pi}{\lambda_1}}\left[\psi_\theta(\tau) - \psi_\theta\left(\tau + \frac{\pi}{\lambda_1}\right)\right]\cos\lambda_1\tau d\tau,$$

$$I_2 = \int_{\frac{\pi}{\lambda_1}}^{2\pi - \frac{\pi}{\lambda_1}}\left[\psi_\theta(\tau) - \psi_\theta\left(\tau + \frac{\pi}{\lambda_1}\right)\right]\cos\lambda_1\tau d\tau,$$

$$I_3 = \int_{2\pi - \frac{\pi}{\lambda_1}}^{2\pi}\left[\psi_\theta(\tau) - \psi_\theta\left(\tau + \frac{\pi}{\lambda_1}\right)\right]\cos\lambda_1\tau d\tau.$$

Since $\left(\frac{2}{\pi}\right)\tau \le \sin\tau \le \tau,\ \tau \in \left[0, \frac{\pi}{2}\right]$, for the function $\psi_\theta(\tau)$ we have

$$\left|\psi_\theta(\tau)\right| \le \frac{2^\alpha\pi\left|\sin\theta\right|^\alpha\left|\sin\tau\right|^\alpha}{\tau}.$$

Then for the integral I_1 the following estimation holds:

$$\left|I_1\right| \le 2^\alpha\pi\left|\sin\theta\right|^\alpha\left(\int_0^{\frac{\pi}{\lambda_1}}\tau^{\alpha - 1}d\tau + \int_0^{\frac{\pi}{\lambda_1}}\left(\tau + \frac{\pi}{\lambda_1}\right)^{\alpha - 1}d\tau\right) =$$

$$= \frac{2^{\alpha}\pi|\sin\theta|^{\alpha}}{\alpha}\left(\left(\frac{\pi}{\lambda_1}\right)^{\alpha} + \left(\frac{2\pi}{\lambda_1}\right)^{\alpha} - \left(\frac{\pi}{\lambda_1}\right)^{\alpha}\right) =$$

$$= \frac{2^{2\alpha}\pi^{1+\alpha}|\sin\theta|^{\alpha}}{\alpha\lambda_1^{\alpha}}, \ \alpha \in (0, 1], \ \lambda_1 = n + \frac{1}{2}. \tag{11}$$

Substituting $\tau \mapsto 2\pi - \tau$ in the integral I_3 and applying the same considerations we obtain

$$|I_3| \le \frac{2^{2\alpha}\pi^{1+\alpha}|\sin\theta|^{\alpha}}{\alpha\lambda_1^{\alpha}}, \ \alpha \in (0, 1], \ \lambda_1 = n + \frac{1}{2}. \tag{12}$$

Now we pay attention to the integral I_2. Since the integrand is 2π-periodic, estimations of the integrals over the segments $\left[\frac{\pi}{\lambda_1}, \pi\right]$ and $\left[\pi, 2\pi - \frac{\pi}{\lambda_1}\right]$ coincide. We have

$$\int_{\frac{\pi}{\lambda_1}}^{\pi}\left[\psi_\theta(\tau) - \psi_\theta\left(\tau + \frac{\pi}{\lambda_1}\right)\right]\cos\lambda_1\tau d\tau = I_4 + I_5, \tag{13}$$

where

$$I_4 = \int_{\frac{\pi}{\lambda_1}}^{\pi} \frac{f(\cos(\theta+\tau)) - f(\cos(\theta-\tau)) - f\left(\cos\left(\theta+\tau+\frac{\pi}{\lambda_1}\right)\right) + f\left(\cos\left(\theta-\tau-\frac{\pi}{\lambda_1}\right)\right)}{\sin\frac{\tau+\frac{\pi}{\lambda_1}}{2}}\cos\lambda_1\tau d\tau,$$

$$I_5 = \int_{\frac{\pi}{\lambda_1}}^{\pi}\left[f(\cos(\theta+\tau)) - f(\cos(\theta-\tau))\right]\left[\frac{1}{\sin\frac{\tau}{2}} - \frac{1}{\sin\frac{\tau+\frac{\pi}{\lambda_1}}{2}}\right]\cos\lambda_1\tau d\tau.$$

Taking into account that $f \in H^{(\alpha)}[-1, 1]$, $0 < \alpha \le 1$, we obtain

$$\left|f(\cos(\theta\pm\tau)) - f\left(\cos\left(\theta\pm\tau\pm\frac{\pi}{\lambda_1}\right)\right)\right| \le$$

$$\le \left|2\sin\left(\theta\pm\tau\pm\frac{\pi}{2\lambda_1}\right)\sin\frac{\pi}{2\lambda_1}\right|^{\alpha} = \left|\sin(\theta\pm\tau)\sin\frac{\pi}{\lambda_1} \pm 2\cos(\theta\pm\tau)\sin^2\frac{\pi}{2\lambda_1}\right|^{\alpha} =$$

$$= \left|\sin\theta\cos\tau\sin\frac{\pi}{\lambda_1} \pm \cos\theta\sin\tau\sin\frac{\pi}{\lambda_1} + 2\cos(\theta\pm\tau)\sin^2\frac{\pi}{2\lambda_1}\right|^{\alpha} \le$$

$$\le |\sin\theta|^{\alpha}\sin^{\alpha}\frac{\pi}{\lambda_1} + |\cos\theta|^{\alpha}\tau^{\alpha}\sin^{\alpha}\frac{\pi}{\lambda_1} + 2^{\alpha}\sin^{2\alpha}\frac{\pi}{2\lambda_1}. \tag{14}$$

Therefore, for the integral I_4 the following estimation holds:

$$|I_4| \le 2\pi\left(|\sin\theta|^{\alpha}\left(\frac{\pi}{\lambda_1}\right)^{\alpha} + 2^{\alpha}\left(\frac{\pi}{2\lambda_1}\right)^{2\alpha}\right)\int_{\frac{\pi}{\lambda_1}}^{\pi}\frac{d\tau}{\tau} + 2\pi|\cos\theta|^{\alpha}\left(\frac{\pi}{\lambda_1}\right)^{\alpha}\int_{\frac{\pi}{\lambda_1}}^{\pi}\tau^{\alpha-1}d\tau \le$$

$$\le 2\pi^{1+\alpha}|\sin\theta|^{\alpha}\frac{\ln\lambda_1}{\lambda_1^{\alpha}} + \frac{2\pi^{2\alpha+1}|\cos\theta|^{\alpha}}{\alpha\lambda_1^{\alpha}} + 2^{1-\alpha}\pi^{1+2\alpha}\frac{\ln\lambda_1}{\lambda_1^{2\alpha}}. \tag{15}$$

Consider the integral I_5. Since

$$\left|f\left(\cos(\theta+\tau)\right)-f\left(\cos(\theta-\tau)\right)\right|\le 2^{\alpha}\left|\sin\theta\right|^{\alpha}\tau^{\alpha},$$

and

$$\frac{1}{\sin\frac{\tau}{2}}-\frac{1}{\sin\frac{\tau+\frac{\pi}{\lambda_1}}{2}}\le\frac{\pi^3}{2\lambda_1\tau^2},$$

we get

$$\left|I_5\right|\le\frac{2^{\alpha-1}\pi^3}{\lambda_1}\left|\sin\theta\right|^{\alpha}\int\limits_{\frac{\pi}{\lambda_1}}^{\pi}\tau^{\alpha-2}\,d\tau.$$

The last relation leads to the estimation

$$\left|I_5\right|\le\begin{cases}\dfrac{\pi^{2+\alpha}\left|\sin\theta\right|^{\alpha}}{2^{1-\alpha}(1-\alpha)\lambda_1^{\alpha}}, & \alpha\in(0,1),\\ \pi^3\left|\sin\theta\right|\dfrac{\ln\lambda_1}{\lambda_1}, & \alpha=1.\end{cases}\tag{16}$$

Substituting (15) and (16) into (13), we obtain for the integral I_2

$$\left|I_2\right|\le\begin{cases}8\pi^{1+\alpha}\left|\sin\theta\right|^{\alpha}\dfrac{\ln\lambda_1}{\lambda_1^{\alpha}}+\dfrac{4\pi^{2\alpha+1}\left|\cos\theta\right|^{\alpha}}{\alpha\lambda_1^{\alpha}}+2^{2-\alpha}\pi^{1+2\alpha}\dfrac{\ln\lambda_1}{\lambda_1^{2\alpha}}, & \alpha\in(0,1),\\ 4\pi^3\left|\sin\theta\right|\dfrac{\ln\lambda_1}{\lambda_1}+\dfrac{4\pi^3\left|\cos\theta\right|}{\lambda_1}+2\pi^3\dfrac{\ln\lambda_1}{\lambda_1^2}, & \alpha=1.\end{cases}\tag{17}$$

Using inequalities (11), (12) and (17), it follows from (10)

$$\left|\hat{\varepsilon}_n(f,x)\right|\le\begin{cases}\pi^{\alpha}\left|\sin\theta\right|^{\alpha}\dfrac{\ln\lambda_1}{\lambda_1^{\alpha}}+\dfrac{\pi^{\alpha}\left|\sin\theta\right|^{\alpha}}{2^{2-2\alpha}\alpha\lambda_1^{\alpha}}+\dfrac{\pi^{2\alpha}\left|\cos\theta\right|^{\alpha}}{2\alpha\lambda_1^{\alpha}}+\dfrac{\pi^{2\alpha}\ln\lambda_1}{2^{1+\alpha}\lambda_1^{2\alpha}}, & \alpha\in(0,1),\\ \pi^2\left|\sin\theta\right|\dfrac{\ln\lambda_1}{2\lambda_1}+\dfrac{\pi\left|\sin\theta\right|}{\lambda_1}+\dfrac{\pi^2\left|\cos\theta\right|}{2\lambda_1}+\pi^2\dfrac{\ln\lambda_1}{4\lambda_1^2}, & \alpha=1.\end{cases}\tag{18}$$

Finally, we choose n that $\lambda_1=n+\frac{1}{2}$ satisfies the following conditions: $\ln\lambda_1>\frac{1}{\alpha 2^{2-2\alpha}}$, $\alpha\in(0,1)$, and $\ln\lambda_1>\frac{\pi}{2}$, $\alpha=1$. Then, taking into account that $x=\cos\theta$, from estimation (18), we get (7). Theorem 1 is proved.

Approximation of functions $\tilde{f}(x)$

Let

$$\tilde{\varepsilon}_n(f,x)=\tilde{f}(x)-s_n\left(\tilde{f},x\right),\ x\in[-1,1],$$

$$\tilde{\varepsilon}_n(f)=\left\|\tilde{f}(x)-s_n\left(\tilde{f},x\right)\right\|_{C[-1,1]},\ n\in\mathbb{N},$$

see also (5) and (6).

Theorem 2. *For approximation of function (3) with the density $f\in W^1H^{(\alpha)}[-1,1]$, $\alpha\in(0,1]$, by partial sums of its Fourier – Chebyshev series, the following integral representation holds for $n\in\mathbb{N}$:*

$$\tilde{\varepsilon}_n(f,x)=\frac{-1}{4\pi}\int\limits_{-\pi}^{\pi}\left[f'\left(\cos(\theta+\tau)\right)\sin(\theta+\tau)-f'\left(\cos(\theta-\tau)\right)\sin(\theta-\tau)\right]\hat{D}_n(\tau)d\tau,\tag{19}$$

where $x=\cos\theta$ and $\hat{D}_n(\tau)$ is from (8).

P r o o f. Let us use the integral representation of approximation (9). We assume that the density $f\in C^{(1)}[-1,1]$ and $f'\in H^{(\alpha)}[-1,1]$, $\alpha\in(0,1]$. Therefore, the integral in the right-hand side can be differentiated with respect to $x\in(-1,1)$, $x=\cos\theta$. Then

$$\hat{\varepsilon}_n'(f, x) = \frac{-1}{2\pi\sqrt{1-x^2}} \int_0^{\pi} \left[f'(\cos(\theta+\tau))\sin(\theta+\tau) - f'(\cos(\theta-\tau))\sin(\theta-\tau) \right] \hat{D}_n(\tau)\, d\tau.$$

In addition, from the previous considerations it follows that

$$\tilde{\varepsilon}_n(f, x) = \sqrt{1-x^2}\, \hat{\varepsilon}_n'(f, x), \; x \in (-1, 1).$$

Since the integrand is 2π-periodic, we get representation (19). Theorem 2 is proved.

Now we apply theorem 2 for the approximation of functions $\tilde{f}$, defined by (3), whose density f satisfies the condition $f' \in H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, on the interval $[-1, 1]$.

Theorem 3. *For approximation of function (3) with the density* $f \in W^1 H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, *by partial sums of its Fourier – Chebyshev series, the following estimation holds for sufficiently large n:*

$$|\tilde{\varepsilon}_n(f, x)| \leq \begin{cases} 4\pi^{\alpha}\left(1-x^2\right)^{\frac{1+\alpha}{2}} \dfrac{\ln n}{n^{\alpha}} + \dfrac{c_{\alpha}(x)}{n^{\alpha}} + \dfrac{\pi^{2\alpha}|x|}{2^{\alpha} n^{2\alpha}}, \; \alpha \in (0, 1), \\ 2\pi^2\left(1-x^2\right)\dfrac{\ln n}{n} + \dfrac{c_1(x)}{n^{\alpha}} + \dfrac{\pi^2|x|}{2n^2}, \; \alpha = 1, \end{cases} \tag{20}$$

where

$$c_{\alpha}(x) = \frac{\pi^{2\alpha}\sqrt{1-x^2}\,|x|^{\alpha}}{2\alpha} + \pi^{\alpha}\left(\sqrt{1-x^2}\right)^{\alpha}|x| + \frac{2^{\alpha}\pi^{2\alpha}|x|^{1+\alpha}}{1+\alpha}.$$

P r o o f. We get from (19)

$$\tilde{\varepsilon}_n(f, x) = \frac{1}{4\pi}\left[\sin\theta I_6 + 2\cos\theta I_7\right], \; x = \cos\theta, \; x \in [-1, 1], \tag{21}$$

where

$$I_6 = \int_0^{2\pi} \left[f'(\cos(\theta+\tau)) - f'(\cos(\theta-\tau)) \right] \cos\tau \frac{\cos\left(n+\frac{1}{2}\right)\tau}{\sin\frac{\tau}{2}} d\tau,$$

$$I_7 = \int_0^{2\pi} \left[f'(\cos(\theta+\tau)) + f'(\cos(\theta-\tau)) \right] \cos\frac{\tau}{2}\cos\left(n+\frac{1}{2}\right)\tau\, d\tau.$$

Let us study each of these integrals. To estimate the integral I_6 we use the same idea as for the proof of theorem 1. So, for sufficiently large n we have

$$|I_6| \leq \begin{cases} 8\pi^{1+\alpha}|\sin\theta|^{\alpha}\dfrac{\ln n}{n^{\alpha}} + \dfrac{2\pi^{2\alpha+1}|\cos\theta|^{\alpha}}{\alpha n^{\alpha}} + 2^{1-\alpha}\pi^{1+2\alpha}\dfrac{\ln n}{n^{2\alpha}}, \; \alpha \in (0, 1), \\ 4\pi^3|\sin\theta|\dfrac{\ln n}{n} + \dfrac{2\pi^3|\cos\theta|}{n} + \pi^3\dfrac{\ln n}{n^2}, \; \alpha = 1. \end{cases} \tag{22}$$

We represent the integral I_7 as follows:

$$I_7 = \frac{1}{2}\left[I_{71} + I_{72}\right], \tag{23}$$

where

$$I_{71} = \int_0^{2\pi} \left[f'(\cos(\theta+\tau)) + f'(\cos(\theta-\tau)) \right] \cos n\tau\, d\tau,$$

$$I_{72} = \int_0^{2\pi} \left[f'(\cos(\theta+\tau)) + f'(\cos(\theta-\tau)) \right] \cos(n+1)\tau\, d\tau.$$

We estimate integrals I_{71} and I_{72} similarly. Bearing in mind the properties of the integrand, we have for the integral I_{71}

$$I_{71} = \frac{1}{2}\int_0^{2\pi} \Big(-\left[f'(\cos(\theta+\tau)) + f'(\cos(\theta-\tau)) \right] -$$

$$-\left[f'\left(\cos\left(\theta+\tau+\frac{\pi}{n}\right)\right)+f'\left(\cos\left(\theta-\tau-\frac{\pi}{n}\right)\right)\right]\Bigg)\cos n\tau d\tau.$$

Taking into account estimation (14), we obtain from the latter representation

$$|I_{71}|\leq\frac{2\pi^{1+\alpha}|\sin\theta|^{\alpha}}{n^{\alpha}}+\frac{2^{1+\alpha}\pi^{1+2\alpha}|\cos\theta|^{\alpha}}{(1+\alpha)n^{\alpha}}+\frac{2^{1-\alpha}\pi^{1+2\alpha}}{n^{2\alpha}}.$$

Similarly,

$$|I_{72}|\leq\frac{2\pi^{1+\alpha}|\sin\theta|^{\alpha}}{(n+1)^{\alpha}}+\frac{2^{1+\alpha}\pi^{1+2\alpha}|\cos\theta|^{\alpha}}{(1+\alpha)(n+1)^{\alpha}}+\frac{2^{1-\alpha}\pi^{1+2\alpha}}{(n+1)^{2\alpha}}.$$

Substituting the estimations for integrals I_{71} and I_{72} into (23), we get

$$|I_7|\leq\frac{2\pi^{1+\alpha}|\sin\theta|^{\alpha}}{n^{\alpha}}+\frac{2^{1+\alpha}\pi^{1+2\alpha}|\cos\theta|^{\alpha}}{(1+\alpha)n^{\alpha}}+\frac{2^{1-\alpha}\pi^{1+2\alpha}}{n^{2\alpha}},\ x=\cos\theta,\ x\in[-1,1]. \tag{24}$$

Let us return to the proof of theorem 3. Estimation (20) follows directly from (21), if we use inequalities (22) and (24). Theorem 3 is proved.

Remark. It is important to note that approximation of functions of classes (3) with the density $f\in W^1H^{(\alpha)}[-1,1]$, $\alpha\in(0,1]$, by partial sums of their Fourier – Chebyshev series depends on the location of a point on the segment. Moreover, the approximation at the endpoints of the segment has a higher rate of decrease than on the whole segment.

Approximation of the function $\tilde{f}(x)$ with the density $|t|^s$, $s>1$

In paper [20], the authors studied the approximations of conjugate functions (1) with the density $f(t)=|t|^s$, $s>1$, by partial sums of their conjugate Fourier – Chebyshev series. Here we consider a similar problem on classes of functions (3). Since there is a certain relationship between the elements of these functional classes, it is interesting to compare the orders of their approximations and the corresponding constants. For these approximations we have (see also (5) and (6))

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right)=\tilde{f}(x)-s_{2n}\left(\tilde{f},\ x\right),\ x\in[-1,1], \tag{25}$$

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s\right)=\left\|\tilde{f}(x)-s_{2n}\left(\tilde{f},\ x\right)\right\|_{C[-1,1]},\ n\in\mathbb{N}. \tag{26}$$

Theorem 4. *Approximation of the function* $\tilde{f}(x)$ *with the density* $|t|^s$, $s>1$, *on the segment* $[-1,1]$ *by partial sums of its Fourier – Chebyshev series satisfies the following properties.*

1. Integral representation:

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right)=\frac{(-1)^{n+1}}{2^{s-2}\pi}\sin\frac{\pi s}{2}\int_0^1\left(1-t^2\right)^s t^{2n+3-s}\frac{p_6(t,x)}{\left(1+2t^2T_2(x)+t^4\right)^2}dt,\ x\in[-1,1], \tag{27}$$

where

$$p_6(t,x)=nt^6T_{2n}(x)+t^4\left(2nT_{2n+2}(x)+(n+1)T_{2n-2}(x)\right)+$$
$$+t^2\left(nT_{2n+4}(x)+2(n+1)T_{2n}(x)\right)+(n+1)T_{2n+2}(x), \tag{28}$$

$T_{2n}(\cdot)$ *are Chebyshev polynomials of the first kind.*

2. Estimations for pointwise approximation:

$$\left|\tilde{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right)\right|\leq\frac{1}{2^{s-2}\pi}\left|\sin\frac{\pi s}{2}\right|\left|\int_0^1\left(1-t^2\right)^{s-2}t^{2n+3-s}\frac{|p_6(t,x)|}{\left(1-t^2\right)^2}dt\right|,\ x\in[-1,1]. \tag{29}$$

3. Representation for uniform approximation:

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s\right)=\frac{1}{2^{s-2}\pi}\left|\sin\frac{\pi s}{2}\right|\left|\int_0^1\left(1-t^2\right)^{s-2}t^{2n+3-s}\left[(n+1)-nt^2\right]dt,\ s>1,\ n\in\mathbb{N}.\right. \tag{30}$$

4. Asymptotic estimation for uniform approximation:

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s\right) \sim \frac{1}{\pi}\left|\sin\frac{\pi s}{2}\right|\frac{s\Gamma(s-1)}{\left(2(n+1)\right)^{s-1}},\ s\in(1,+\infty)\setminus\mathbb{N},\ n\to\infty, \tag{31}$$

where $\Gamma(\cdot)$ *is a gamma function.*

Estimation (29) is exact. It turns into equality when $x = 0$.

P r o o f. From representation (3) it follows that approximations (25) have the form

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right) = -\sum_{k=n+1}^{+\infty} kc_{2k}T_{2k}(x),\ x\in[-1,1],\ n\in\mathbb{N}, \tag{32}$$

where c_{2k} are polynomial Fourier – Chebyshev coefficients of the function $|x|^s$, $s>0$. It is known (see [31, theorem 5]) that for c_{2k} the following integral representation holds:

$$c_{2k} = \frac{(-1)^{k-1}}{2^{s-2}\pi}\sin\frac{\pi s}{2}\int_0^1\left(1-t^2\right)^s t^{2k+1-s}\,dt,\ k=1,2,\ldots.$$

Applying the Laplace method (see, for example, [29; 30]) for the study of integrals' asymptotic behaviour, we can show that

$$c_{2k} \sim \frac{(-1)^{k-1}}{2^{s-1}\pi}\sin\frac{\pi s}{2}\frac{\Gamma(s+1)}{k^{s+1}},\ s>1,\ k=n+1,\ n+2,\ \ldots,\ k\to\infty,$$

$\Gamma(\cdot)$ is a gamma function. The series

$$\frac{\Gamma(s+1)}{2^{s-1}\pi}\sin\frac{\pi s}{2}\sum_{k=n+1}^{+\infty}\frac{1}{k^s},\ s>1,$$

is a convergent majorant series to the series on the right-hand side of (32), which converges uniformly on the entire segment $[-1,1]$. Then plugging the integral representation for coefficients c_{2k} into (32) and interchanging the summation and the integration, we have

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right) = \frac{1}{2^{s-2}\pi}\sin\frac{\pi s}{2}\int_0^1\left(1-t^2\right)^s t^{1-s}\Phi_n(t,x)\,dt,\ x\in[-1,1],\ n\in\mathbb{N}, \tag{33}$$

where

$$\Phi_n(t,x) = \sum_{k=n+1}^{+\infty}(-1)^k kt^{2k}T_{2k}(x),\ T_{2k}(x) = \cos(2k\arccos x).$$

The following equality follows from the properties of the sum of the geometric series

$$\sum_{k=n+1}^{+\infty}(-1)^k kq^k = (-1)^{n+1}\frac{(n+1)q^{n+1}+nq^{n+2}}{(1+q)^2},\ |q|<1. \tag{34}$$

Assuming here $q = t^2e^{2i\theta}$, $x=\cos\theta$ and separating the real and imaginary parts we obtain

$$\Phi_n(t,x) = (-1)^{n+1}\frac{t^{2n+2}p_6(t,x)}{\left(1+2t^2T_2(x)+t^4\right)^2},$$

where $p_6(t,x)$ is defined in (28).

The integral representation (27) follows from the last relation and (33). Note, that

$$p_6(t,0) = (-1)^n(t-1)^2\left[nt^4+2nt^3-t^2-2(n+1)t-(n+1)\right].$$

In other words, the polynomial $p_6(t,0)$ has $t=1$ as a zero of the second order. Thus, (29) immediately follows from this fact and inequality

$$\sqrt{1+2t^2\cos 2\theta+t^4} \geq 1-t^2,\ t\in[0,1],\ \theta\in\mathbb{R}.$$

The exactness of estimation (29) can be verified directly by substituting $x=0$ into (27).

Relation (30) for the uniform approximations (26) can be easily established with the help of integral representation (33). Indeed, it is obvious that

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s\right) \leq \frac{1}{2^{s-2}\pi}\left|\sin\frac{\pi s}{2}\right|\left|\int_0^1\left(1-t^2\right)^s t^{1-s}\sum_{k=n+1}^{+\infty}kt^{2k}\,dt\right|,\ n\in\mathbb{N}.$$

Using (34), we get

$$\tilde{\varepsilon}_{2n}\left(|\cdot|^s\right) \leq \frac{1}{2^{s-2}\pi}\left|\sin\frac{\pi s}{2}\right|\int_0^1\left(1-t^2\right)^s t^{2n+3-s}\frac{(n+1)-nt^2}{\left(1-t^2\right)^2}dt,\ n\in\mathbb{N}.$$

Then, relation (30) follows from the last inequality and the exactness of estimation (29).

In order to establish asymptotic estimation (31) we study the behaviour of the integral on the right side of (30) when $n \to \infty$. To solve this problem, we again use the Laplace method [29; 30]. We write the integral as follows:

$$\int_0^1\left(1-t^2\right)^{s-2} t^{2n+3-s}\left[(n+1)-nt^2\right]dt = (n+1)J_1 + J_2,\ n\in\mathbb{N}, \tag{35}$$

where

$$J_1 = \int_0^1\left(1-t^2\right)^{s-1} t^{1-s}\mathrm{e}^{2(n+1)\ln t}dt,\ J_2 = \int_0^1\left(1-t^2\right)^{s-2} t^{3-s}\mathrm{e}^{2(n+1)\ln t}dt.$$

The asymptotic behaviour of these integrals is studied similarly. Consider the first of them. The function $S(t) = \ln t$ monotonically decreases for $0 < t < 1$ and reaches its maximal value for $t = 1$. Since $\ln t \sim (t-1)$ and $\left(1-t^2\right)^{s-2} t^{1-s} \sim 2^{s-2}(1-t)^{s-2}$ when $t \to 1$, for sufficiently small $\varepsilon > 0$ and $n \to \infty$ we get

$$J_1 \sim 2^{s-1}\int_{1-\varepsilon}^1 (1-t)^{s-1}\mathrm{e}^{2(n+1)(t-1)}dt.$$

After the substitution $2(n+1)(1-t) \mapsto u,$ we have

$$J_1 \sim \frac{1}{2(n+1)^s}\int_0^{2(n+1)\varepsilon} u^{s-1}\mathrm{e}^{-u}du \sim \frac{\Gamma(s)}{2(n+1)^s},\ n\to\infty.$$

Similarly,

$$J_2 \sim \frac{\Gamma(s-1)}{2(n+1)^{s-1}},\ n\to\infty.$$

Taking into account the obtained results, from (35) we find that

$$\int_0^1\left(1-t^2\right)^{s-2} t^{2n+3-s}\left[(n+1)-nt^2\right]dt \sim \frac{s\Gamma(s-1)}{2(n+1)^{s-1}},\ n\to\infty.$$

The last asymptotic equality, representation (30), and the accuracy of estimation (29) lead us to relation (31). Theorem 4 is proved.

It is interesting to compare asymptotic estimation (31) for uniform approximations of function (3) with the density $|t|^s$, $s > 1$, by partial sums of its Fourier – Chebyshev series with the corresponding uniform estimation for the approximation of function (1) by the conjugate Fourier – Chebyshev series established in [20]:

$$\left|\hat{\varepsilon}_{2n}\left(|\cdot|^s,\ x\right)\right| \leq \frac{|\sin 2u|}{\pi}\left|\sin\frac{\pi s}{2}\right|\frac{s\Gamma(s-1)}{\left(2(n+1)\right)^{s-1}},\ x=\cos u,\ s\in(1,\ +\infty)\setminus 2\mathbb{N},\ n\to\infty.$$

Conclusions

In this paper, approximations on the interval $[-1, 1]$ of conjugate functions (1) with the density $f \in H^{(\alpha)}[-1, 1]$, $\alpha \in (0, 1]$, by the conjugate Fourier – Chebyshev series are studied. Order estimations of the approximation depending on the location of a point on a segment are established. It is noted that the approximation at the ends of the segment has a higher rate of decrease in comparison with the whole segment.

Classes of functions (3), which can be associated in a certain sense with the derivative of conjugate function (1) are introduced. Approximations of functions from these classes by partial sums of the Fourier – Chebyshev series are studied. An integral representation of the approximation is found. In the case when the density $f \in W^1H^{(\alpha)}[-1, 1],\ \alpha \in (0, 1]$, the order estimations of the approximation are established, also depending on the location of the point on the segment. Similarly, approximations at the endpoints of a segment have a higher rate of decrease compared to the whole segment.

The case when the density $f(t)=|t|^s$, $s>1$, is considered. An integral representation of the approximation, estimations for the pointwise and uniform approximation, and an asymptotic estimation for the uniform approximation are obtained. It is noted that the order of uniform approximations of the function under study by partial sums of the Fourier – Chebyshev series and the corresponding conjugate function by conjugate sums coincide.

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