Математическая логика, алгебра и теория чисел

Mathematical logic, algebra and number theory

УДК 512.542

ОБ ОДНОЙ ОТКРЫТОЙ ПРОБЛЕМЕ ТЕОРИИ МОДУЛЯРНЫХ ПОДГРУПП

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Пусть G – конечная группа. Подгруппа A группы G называется модулярной в G, если (i) $\langle X, A \cap Z \rangle = \langle X, A \rangle \cap Z$ для всех $X \leq G$, $Z \leq G$ таких, что $X \leq Z$, и (ii) $\langle A, Y \cap Z \rangle = \langle A, Y \rangle \cap Z$ для всех $Y \leq G$, $Z \leq G$ таких, что $A \leq Z$. Получено описание конечных групп, в которых модулярность является транзитивным отношением, т. е. если A – модулярная

Образец цитирования:
Лю Амин-Мин, Го Вэньбинь, Сафонова ИН, Скиба АН. Об одной открытой проблеме теории модулярных подгрупп. *Журнал Белорусского государственного университета. Математика. Информатика.* 2023;2:28–34 (на англ.).
https://doi.org/10.33581/2520-6508-2023-2-28-34
EDN: WELODI

For citation:
Liu Aming-Ming, Guo Wenbin, Safonova IN, Skiba AN. On an open problem in the theory of modular subgroups. *Journal of the Belarusian State University. Mathematics and Informatics.* 2023;2:28–34.
https://doi.org/10.33581/2520-6508-2023-2-28-34
EDN: WELODI

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подгруппа в K и K – модулярная подгруппа в G, то A – модулярная подгруппа в G. Полученный результат является решением одной из старых задач теории модулярных подгрупп, восходящей к работам А. Фриджерио (1974), И. Циммерман (1989).

Ключевые слова: конечная группа; модулярная подгруппа; субмодулярная подгруппа; M-группа; комплекс Робинсона.

Благодарность. Работа выполнена при поддержке Национального фонда естественных наук Китая (грант № 12171126) и Фонда естественных наук провинции Хайнань Китая (грант № 621RC510). Исследование третьего автора поддержано Министерством образования Республики Беларусь (№ гос. регистрации 20211328).

ON AN OPEN PROBLEM IN THE THEORY OF MODULAR SUBGROUPS

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Let G be a finite group. Then a subgroup A of group G is said to be modular in G if (i) $\langle X, A \cap Z\rangle = \langle X, A\rangle \cap Z$ for all $X \le G$, $Z \le G$ such that $X \le Z$, and (ii) $\langle A, Y \cap Z\rangle = \langle A, Y\rangle \cap Z$ for all $Y \le G$, $Z \le G$ such that $A \le Z$. We obtain a description of finite groups in which modularity is a transitive relation, that is, if A is a modular subgroup of K and K is a modular subgroup of G, then A is a modular subgroup of G. The result obtained is a solution to one of the old problems in the theory of modular subgroups, which goes back to the works of A. Frigerio (1974), I. Zimmermann (1989).

Keywords: finite group; modular subgroup; submodular subgroup; M-group; Robinson complex.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (grant No. 12171126) and the Natural Science Foundation of Hainan Province of China (grant No. 621RC510). The research of the third author was supported by the Ministry of Education of the Republic of Belarus (state registration No. 20211328).

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group: G is said to be an M-group [1, p. 54] if the lattice $L(G)$ of all subgroups of G is modular. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

A subgroup A of G is said to be quasinormal (O. Ore) or permutable (S. E. Stonehewer) in G if A permutes with every subgroup H of G, that is, $AH = HA$; Sylow permutable or S-permutable [2; 3] if A permutes with all Sylow subgroups of G.

Quasinormal and Sylow permutable subgroups have many useful properties. For instance, if A is quasinormal in G, then A is subnormal in G [4], A/A_G is nilpotent [5], $C_G(H/K) = G$ for every chief factor H/K of G between A_G and A^G [6], and, in general, the section A/A_G is not necessarily abelian [7].

Quasinormal subgroups have also a close connection with the so-called modular subgroups.

Recall that a subgroup M of G is said to be modular in G if M is a modular element (in the sense of Kurosh [1, p. 43]) of the lattice $L(G)$, that is, (i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \le G$, $Z \le G$ such that $X \le Z$, and (ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \le G$, $Z \le G$ such that $M \le Z$.

Every quasinormal subgroup is clearly modular in the group. Moreover, the following interesting fact is well known.

Theorem 1 [1, theorem 5.1.1]). *A subgroup A of G is quasinormal in G if and only if A is modular and subnormal in G.*

A group G is said to be a T-group if normality is a transitive relation on G, that is, if H is a normal subgroup of K and K is a normal subgroup of G, then H is a normal subgroup of G. In other words, the group G is a T-group if every subnormal subgroup of G is normal in G.

The description of T-groups was first obtained by W. Gaschütz [8] for the soluble case and by D. J. S. Robinson [9] for the general case.

Works [8; 9] aroused great interest in the further study of T-groups and groups in which some conditions of generalised normality are transitive (PT-groups, i. e. groups in which quasinormality is transitive; PST-groups, i. e. groups in which Sylow permutability is transitive, etc.) [2, chapter 2].

However, the following interesting problem still remains open.

Question 1. What is the structure of MT-groups, i. e. groups G in which modularity is a transitive relation on G, that is, if H is a modular subgroup of K and K is a modular subgroup of G, then H is a modular subgroup of G?

Such a problem was first raised in paper [10], where the following theorem was proved, which gives a complete answer to the problem for the soluble case.

Theorem 2 [10]. *A soluble group is an MT-group if and only if G is a group with modular lattice of all subgroups* $L(G)$.

New proof of theorem 1 was obtained in paper [11].

Our main goal here is to give an answer to question 1 for the insoluble case.

Before continuing, we give a few definition.

Definition 1. We say that $(D, Z(D); U_1, \ldots, U_k)$ is a Robinson complex of G if the following conditions hold: (i) $D \neq 1$ is a normal perfect subgroup of G, (ii) $D/Z(D) = U_1/Z(D) \times \ldots \times U_k/Z(D)$, where $U_i/Z(D)$ is a simple chief factor of G, and (iii) every chief factor of G below $Z(D)$ is cyclic.

We say, following D. J. S. Robinson [9], that G satisfies:

(1) $\mathbf{N}_p$ if whenever N is a soluble normal subgroup of G, p'-elements of G induce power automorphism in $O_p(G/N)$;

(2) $\mathbf{P}_p$ if whenever N is a soluble normal subgroup of G, every subgroup of $O_p(G/N)$ is quasinormal in Sylow p-subgroups of G/N.

A subgroup A of G is said to be submodular in G if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \ldots \leq A_n = G$$

such that A_{i-1} is a modular subgroup of A_i for all $i = 1, \ldots, n$. Thus, a group G is an MT-group if and only if every of its submodular subgroups is modular.

Remark 1. It is clear that every subnormal subgroup is submodular. On the other hand, in view of Ore's above-mentioned result, G is a PT-group if and only if every its subnormal subgroup is quasinormal. Therefore, every MT-group is a PT-group.

In view of remark 1, the following well-known result partially describes the structure of insoluble MT-groups.

Theorem 3 [9]. *G is a PT-group if and only if G has a normal perfect subgroup D such that: (i) G/D is a soluble PT-group, and (ii) if* $D \neq 1$*, G has a Robinson complex* $(D, Z(D); U_1, \ldots, U_k)$ *and (iii) for any set* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$*, where* $1 \leq r < k$*, G and* $G/U'_{i_1} \cdots U'_{i_r}$ *satisfy* $\mathbf{N}_p$ *for all* $p \in \pi(Z(D))$ *and* $\mathbf{P}_p$ *for all* $p \in \pi(D)$.

Now, recall that G is a non-abelian P-group (see [1, p. 49]) if $G = A \rtimes \langle t \rangle$, where A is an elementary abelian p-group and an element t of prime order $q \neq p$ induces a non-trivial power automorphism on A. In this case we say that G is a P-group of type (p, q).

Definition 2. We say that G satisfies $\mathbf{M}_p$ ($\mathbf{M}_{p, q}$ respectively) if whenever N is a soluble normal subgroup of G and P/N is a normal non-abelian P-subgroup (a normal P-group of type (p, q) respectively) of G/N, every non-subnormal subgroup of P/N is modular in G/N.

In this article we prove the following theorem, which answers question 1 in the general case.

Theorem 4. *A group G is an MT-group if and only if G has a normal perfect subgroup D such that: (i) G/D is an M-group, and (ii) if D ≠ 1, G has a Robinson complex* $(D, Z(D); U_1, \ldots, U_k)$ *and (iii) for any set* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$*, where* $1 \leq r < k$*, G and* $G/U'_{i_1} \cdots U'_{i_r}$ *satisfy* $\mathbf{N}_p$ *for all* $p \in \pi(Z(D))$*,* $\mathbf{P}_p$ *for all* $p \in \pi(D)$ *and* $\mathbf{M}_{p, q}$ *for all pairs* $\{p, q\} \cap \pi(D) \neq \varnothing$.

The following example shows that, in general, a PT-group may not be an MT-group.

Example 1. (i) Let $\alpha : Z(SL(2, 5)) \to Z(SL(2, 7))$ be an isomorphism and let

$$D := SL(2, 5)SL(2, 7) = (SL(2, 5) \times SL(2, 7))/V,$$

where $V = \left\{ \left(a, \left(a^{\alpha}\right)^{-1} \right) \middle| a \in Z(SL(2, 5)) \right\}$, is the direct product of the groups $SL(2, 5)$ and $SL(2, 7)$ with a joint center (see [12, p. 49]). Let $M = (C_7 \rtimes C_3)(C_{13} \rtimes C_3)$ be the direct product of the groups $C_7 \rtimes C_3$ and $C_{13} \rtimes C_3$

with a joint factor group C_3 (see [12, p. 50]), where $C_7 \rtimes C_3$ is a non-abelian group of order 21, and $C_{13} \rtimes C_3$ is a non-abelian group of order 39. Finally, let $G = D \times M$. We show that G satisfies the conditions in theorem 3.

It is clear also that $D = G^{\mathfrak{S}}$ is a soluble residual of G and $M \simeq G/D$ is a soluble PT-group. In view of [12, Kapitel I, Satz 9.10], $D = U_1U_2$ and $U_1 \cap U_2 = Z(D) = \Phi(D)$, where U_i is normal in D, $U_1/Z(D)$ is a simple group of order 60, and $U_2/Z(D)$ is a simple group of order 168. Hence $(D, Z(D); U_1, U_2)$ is a Robinson complex of G, and the subgroup $Z(D)$ has order 2 and $Z(D) \le Z(G)$. Therefore, conditions (i) and (ii) hold for G. It is not difficult to show that for every prime r dividing $|G|$ and for $O_r(G/N)$, where N is a normal soluble subgroup of G, we have $|O_r(G/N)| \in \{1, r\}$, so condition (iii) also holds for G. Therefore, G is a PT-group by theorem 3.

Now we show that G is not an MT-group. First, note that M has a subgroup $T \simeq C_7 \rtimes C_3$ and $|M : T| = 13$. Then T is a maximal subgroup of M and $M/T_M \simeq C_7 \rtimes C_3$. Hence a subgroup L of T of order 3 is modular in T and T is modular in M by [1, lemma 5.1.2], so L is submodular in G. Finally, L is not modular in M by lemma 2 below. Therefore, G is not an MT-group by theorem 4.

(ii) The group $D \times (C_7 \rtimes C_3)$ is an MT-group by theorem 4.

Premilaries

We use $\mathfrak{A}^*$ to denote the class of all abelian groups of squarefree exponent. It is clear that $\mathfrak{A}^*$ is a hereditary formation, $G^{\mathfrak{A}^*}$ is the intersection of all normal subgroups N of G with $G/N \in \mathfrak{A}^*$.

Lemma 1. *Let A, B and N be subgroups of G, where A is submodular in G, and N is normal in G.*

(1) $A \cap B$ is submodular in B.

(2) AN/N is submodular in G/N.

(3) If $N \le K$ and K/N is submodular in G/N, then K is submodular in G.

(4) $A^{\mathfrak{A}^*}$ is subnormal in G.

(5) If $G = U_1 \times \ldots \times U_k$, where U_i is a simple non-abelian group, then A is normal in G.

P r o o f. Statements (1)–(4) are proved in work [11].

(5) Let $E = U_i A$. Then A is submodular in E by statement (1), so there is a subgroup chain

$$A = E_0 < E_1 < \ldots < E_{t-1} < E_t = E$$

such that E_{i-1} is a maximal modular subgroup of E_i for all $i = 1, \ldots, t$ and for $M = E_{t-1}$ we have $M = A(M \cap U_i)$ and, by [1, lemma 5.1.2], either $M = E_{t-1}$ is a maximal normal subgroup of E or M is a maximal subgroup of E such that E/M_E is a non-abelian group of order qr for primes q and r. In the former case we have $M \cap U_i = 1$, so $A = M$ is normal in E. The second case is impossible since E has no a quotient of order qr. Therefore, $U_i \le N_G(A)$ for all i, so $G \le N_G(A)$. Hence we have statement (5).

The lemma is proved.

Lemma 2 [1, lemma 5.1.9]. *Let M be a modular subgroup of G of prime power order. If M is not quasinormal in G, then $G/M_G = M^G/M_G \times K/M_G$, where M^G/M_G is a non-abelian P-group of order prime to $|K/M_G|$.*

Recall that a group G is said to be an SC-group if every chief factor of G is simple [9].

Lemma 3. *Let G be a non-soluble SC-group and suppose that G has a Robinson complex $(D, Z(D); U_1, \ldots, U_k)$, where $D = G^{\mathfrak{S}} = G^{\mathfrak{U}}$. Let U be a submodular non-modular subgroup of G of minimal order.*

(1) If UU_i'/U_i' is modular in G/U_i' for all $i = 1, \ldots, k$, then U is supersoluble.

(2) If U is supersoluble and UL/L is modular in G/L for all non-trivial nilpotent normal subgroups L of G, then U is a cyclic p-group for some prime p.

P r o o f. Suppose that this lemma is false and let G be a counterexample of minimal order.

(1) Assume statement (1) is false. Suppose that $U \cap D \le Z(D)$. Then every chief factor of U below $U \cap Z(D) = U \cap D$ is cyclic and, also, $UD/D \simeq U/(U \cap D)$ is supersoluble. Hence U is supersoluble, a contradiction. Therefore, $U \cap D \not\le Z(D)$. Moreover, statements (1) and (2) of lemma 1 imply that $(U \cap D)Z(D)/Z(D)$ is submodular in $D/Z(D)$ and so $(U \cap D)Z(D)/Z(D)$ is a non-trivial normal subgroup of $D/Z(D)$ by statement (5) of lemma 1.

Hence for some i we have $U_i/Z(D) \le (U \cap D)Z(D)/Z(D)$, so $U_i \le (U \cap D)Z(D)$. But then $U_i' \le \le ((U \cap D)Z(D))' \le U \cap D$.

By hypothesis, $UU_i'/U_i' = U/U_i'$ is modular in G/U_i' and so U is modular in G by [1, p. 201, property (4)], a contradiction. Therefore, statement (1) holds.

(2) Assume statement (2) is false. Let $N = U^{\mathfrak{N}}$ be the nilpotent residual of U. Then $N < U$ since U supersoluble, so N is modular in G. It is clear that every proper subgroup S of U with $N \leq S$ is submodular in G, so the minimality of U implies that S is modular in G. Therefore, if U has at least two distinct maximal subgroups S and W such that $N \leq S \cap W$, then $U = \langle S, W \rangle$ is modular in G by [1, p. 201, property (5)], contrary to our assumption on U. Hence U/N is a cyclic p-group for some prime p and $N \neq 1$ since U is not cyclic.

Now we show that U is a PT-group. Let S be a proper subnormal subgroup of U. Then S is submodular in G since U is submodular in G, so S is modular in G and hence S is quasinormal in U by theorem 1. Therefore, U is a soluble PT-group, so $N = U^{\mathfrak{N}} = U'$ is a Hall abelian subgroup of U and every subgroup of N is normal in U by [2, theorem 2.1.11]. Then $N \leq U^{\mathfrak{A}^*}$ and so $U^{\mathfrak{A}^*} = NV$, where V is a maximal subgroup of a Sylow p-subgroup $P \simeq U/N$ of U. Then NV is modular in G and NV is subnormal in G by statement (4) of lemma 1. Therefore, NV is quasinormal in G by theorem 1. Assume that for some minimal normal subgroup R of G we have $R \leq (NV)_G$. Then U/R is a modular in G/R by hypothesis, so U is modular in G, a contradiction. Therefore, $(NV)_G = 1$, so NV is nilpotent by [2, corollary 1.5.6] and then V is normal in U.

Some maximal subgroup W of N is normal in U with $|N : W| = q$. Then $S = WP$ is a maximal subgroup of U such that U/S_U is a non-abelian group of order pq. Hence S is modular in U by [1, lemma 5.1.2], so S is modular in G. It follows that $U = NS$ is modular in G, a contradiction. Therefore, statement (2) holds.

The lemma is proved.

Lemma 4. *If G is an MT-group, then every quotient G/N of G is also an MT-group.*

P r o o f. Let L/N be submodular subgroup of G/N. Then L is submodular subgroup in G by statement (3) of lemma 1, so L is modular in G and then L/N is modular in G/N by [1, p. 201, property (3)]. Hence G/N is an MT-group.

The lemma is proved.

Lemma 5. *If G is an MT-group, then G/R satisfies* $\mathbf{M}_p$ *for every normal subgroup R of G.*

P r o o f. In view of lemma 4, we can assume without loss of generality that $R = 1$. Let P/N be a normal non-abelian P-subgroup of G/N and let $L/N \leq P/N$. Then L/N is modular in P/N by [1, lemma 2.4.1], so L/N is submodular in G/N and hence L/N is modular in G/N. Therefore, L is modular in G by [1, p. 201, property (4)]. Hence G satisfies $\mathbf{M}_p$.

Lemma 6 [2, remark 1.6.8]. *Suppose that G has a Robinson complex $(D, Z(D); U_1, \ldots, U_k)$ and let N be a normal subgroup of G.*

(1) If $N = U_i'$ and $k \neq 1$, then $Z(D/N) = U_i/N = Z(D)N/N$ and

$$\left(D/N, Z(D/N); U_1N/N, \ldots, U_{i-1}N/N, U_{i+1}N/N, \ldots, U_kN/N\right)$$

is a Robinson complex of G/N.

(2) If N is nilpotent, then $Z(DN/N) = Z(D)N/N$ and

$$\left(DN/N, Z(DN/N); U_1N/N, \ldots, U_kN/N\right)$$

is a Robinson complex of G/N.

Proposition 1. *Suppose that a group G is a soluble PT-group and let p be a prime. If every submodular p-subgroup of G is modular in G, then every p-subgroup of G is modular in G. In particular, if every submodular subgroup of a supersoluble group G is modular in G, then G is an M-group.*

P r o o f. Assume that this proposition is false and let G be a counterexample of minimal order. Then, by [2, theorem 2.1.11], the following conditions are satisfied: the nilpotent residual D of G is an Hall abelian subgroup of odd order, G acts by conjugation on D as group power automorphisms, and every subgroup of G/D is quasinormal in G/D. Let M be a complement to D in G.

Let U be a non-modular p-subgroup of G of minimal order. Then U is not submodular and every maximal subgroup of U is modular in G, so U is a cyclic group by [1, p. 201, property (5)]. Let V be the maximal subgroup of U. Then $V \neq 1$ since every subgroup of prime order of a supersoluble group is submodular by [11, lemma 6].

We can assume without loss of generality that $U \leq M$ since M is a Hall subgroup of G.

(1) *If R is a normal p-subgroup of G, then every p-subgroup of G containing R is modular in G. In particular, $U_G = 1$ and so $U \cap D = 1$.*

Let L/R be a submodular p-subgroup of G/R. Then L is a submodular p-subgroup of G by [11, lemma 1 (iii)], so L is modular in G by hypothesis. Hence L/R is modular in G/R by [1, p. 201, property (4)]. Thus, the hypothesis holds for G/R. Therefore, every p-subgroup S/R of G/R is modular in G/R by the choice of G, so S is modular in G by [1, p. 201, property (4)].

In view of claim (1) we can assume without loss of generality that $U \leq M$ since M is a Hall subgroup of G.

(2) *If K is a proper submodular subgroup of G, then every p-subgroup L of K is modular in G, so every proper subgroup of G containing U is not submodular in G.*

The subgroup K is a PT-group by [2, corollary 2.11] and if S is a submodular subgroup of K, then S is submodular in G and so S is modular in G. Hence S is modular in K. Therefore, the hypothesis holds for K, so every p-subgroup L of K is modular in K by the choice of G. Hence L is modular in G by hypothesis.

(3) $DU = G$ (this follows from claim (2) and the fact that every subgroup of G containing D is subnormal in G).

(4) *V is not subnormal in G.*

Assume that V is subnormal in G. Then V is quasinormal in G by theorem 1 since V is modular in G. Therefore, $1 < V \leq R = O_p\left(Z_\infty\left(G\right)\right)$ by [2, corollary 1.5.6] since $V_G = 1 = U_G$ by claim (1). But $R \leq U$ by claim (3), hence $R = V = 1$ and so $\left|U\right| = p$, a contradiction. Hence we have claim (4).

(5) $G = V^G \times K$, *where* V^G *is a non-abelian P-group of order prime to* $\left|K\right|$ (since $V_G = 1$, this follows from claim (4) and lemma 2).

From claim (5) it follows that $U \leq V^G$, so U is submodular in G by [1, theorem 2.4.4]). This final contradiction completes the proof of the result.

The proposition is proved.

Outline of the proof of theorem 4

First assume that G is an MT-group. Then G is a PT-group and every quotient G/N is an MT-group by lemma 4. Moreover, by theorem 3, G has a normal perfect subgroup D such that: G/D is a soluble PT-group, and if $D \neq 1$, G has a Robinson complex $\left(D, Z\left(D\right); U_1, \ldots, U_k\right)$ and for any set $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, where $1 \leq r < k$, G and $G/U'_{i_1} \cdots U'_{i_r}$ satisfy $\mathbf{N}_p$ for all $p \in \pi\left(Z\left(D\right)\right)$ and $\mathbf{P}_p$ for all $p \in \pi\left(D\right)$. In view of lemma 5, G and $G/U'_{i_1} \cdots U'_{i_r}$ satisfy $\mathbf{M}_{p,\,q}$ for all pairs $\{p, q\} \cap \pi\left(D\right) \neq \varnothing$.

In view of [2, theorem 2.1.11], G/D is a supersoluble PT-group, and if U/D is a submodular subgroup of G/D, the U is submodular in G by statement (3) of lemma 1, so U is modular in G by hypothesis and hence U/D is modular in G/D by [1, p. 201, property (4)]. Therefore, G/D is an M-group by proposition 1.

Thus, the necessity of the condition of the theorem holds.

Now, assume, arguing by contradiction, that G is a non-MT-group of minimal order satisfying conditions (i), (ii) and (iii).

Then $D \neq 1$ and G has a submodular subgroup U such that U is not modular in G but every submodular subgroup U_0 of G with $U_0 < U$ is modular in G. Let $Z = Z\left(D\right)$. Then $Z \leq \Phi\left(U_i\right) \leq \Phi\left(D\right)$ since D/Z is perfect.

Using lemmas 1–5 and proposition 1, we can show that:

(i) G has a normal subgroup C_q of order q for some $q \in \pi\left(Z\left(D\right)\right)$;

(ii) $E := C_q U = C_q \times U$ is not subnormal in G and, also, $E_G = C_q$.

Hence $G/E_G = E^G/E_G \times K/E_G = C_q U^G/C_q \times K/C_q$, where $E^G/E_G = C_q U^G/C_q \simeq U^G/\left(C_q \cap U^G\right)$ is a non-abelian P-group of order prime to $\left|K/C_G\right|$ by lemma 2. Hence G is a π-decomposable group, where $\pi = \pi\left(U^G/\left(C_q \cap U^G\right)\right)$.

Then D/C_q is π-decomposable. But $C_q \leq \Phi(D)$, so q divides $\left|D/C_q\right|$. Hence q does not divide $\left|C_q U^G/C_q\right|$.

If $C_q \cap U^G = 1$, then $U^G \simeq C_q U^G/C_q$ is a non-abelian P-group, contrary, so $C_q \leq U^G$. Then C_q is a Sylow q-subgroup of U^G. Hence $U^G = C_q \rtimes \left(R \rtimes U\right)$, where $R \rtimes U \simeq U^G/C_q$ is a non-abelian P-group. Let $C = C_{U^G}\left(C_q\right)$. Then $U \leq C$ and so, by [1, lemma 2.2.2], $R \rtimes U = U^{R \rtimes U} \leq C$. Hence $C_q \leq Z\left(U^G\right)$. Therefore, $U^G = C_q \times \left(R \rtimes U\right)$, where $R \rtimes U$ is characteristic in U^G and so it is normal in G. But then $U^G = R \rtimes U \neq C_q \rtimes \left(R \rtimes U\right)$, a contradiction.

The theorem is proved.

Note that another type of generalised T-groups was considered in paper [13].

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Received 27.02.2023 / revised 03.05.2023 / accepted 02.06.2023.