# Математическая логика, алгебра и теория чисел

# Mathematical logic, algebra and number theory

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## ОПИСАНИЕ ЛОКАЛЬНЫХ ОПЕРАТОРОВ УМНОЖЕНИЯ НА КОНЕЧНОМЕРНЫХ АССОЦИАТИВНЫХ АЛГЕБРАХ

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В 2020 г. Ф. Арзикулов и Н. Умрзаков ввели и изучили понятие (линейного) локального оператора умножения на ассоциативных алгебрах. Они доказали, что каждый локальный оператор левого (правого) умножения на кольце матриц над телом является оператором левого (правого соответственно) умножения. Настоящая статья посвящена (линейным) локальным операторам слабого левого (правого) умножения на 5-мерных естественным образом градуированных 2-филиформных нерасщепляемых ассоциативных алгебрах. Разработан алгоритм получения общего вида матриц операторов слабых левых (правых) умножений на 5-мерных естественным образом градуированных 2-филиформных нерасщепляемых ассоциативных алгебрах $\lambda_1^5$ и $\lambda_2^5$, построенных И. Каримжановым и М. Ладрой. Алгоритм получения общего вида матриц локальных операторов слабых левых (правых) умножений на алгебрах $\lambda_1^5$ и $\lambda_2^5$ также разработан. Показано, что ассоциативные алгебры $\lambda_1^5$ и $\lambda_2^5$ имеют локальные операторы слабого левого (правого) умножения, которые не являются операторами слабого левого (правого соответственно) умножения.

***Ключевые слова:*** ассоциативная алгебра; оператор левого (правого) умножения; дифференцирование; локальное дифференцирование; локальный оператор левого (правого) умножения.

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# DESCRIPTION OF LOCAL MULTIPLIERS ON FINITE-DIMENSIONAL ASSOCIATIVE ALGEBRAS

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In 2020 F. Arzikulov and N. Umrzaqov introduced the concept of a (linear) local multiplier. They proved that every local left (right) multiplier on the matrix ring over a division ring is a left (right, respectively) multiplier. This paper is devoted to (linear) local weak left (right) multipliers on 5-dimensional naturally graded 2-filiform non-split associative algebras. An algorithm for obtaining a common form of the matrices of the weak left (right) multipliers on the 5-dimensional naturally graded 2-filiform non-split associative algebras $\lambda_1^5$ and $\lambda_2^5$, constructed by I. Karimjanov and M. Ladra, is developed. An algorithm for obtaining a general form of the matrices of the local weak left (right) multipliers on the algebras $\lambda_1^5$ and $\lambda_2^5$ is also developed. It turns out that the associative algebras $\lambda_1^5$ and $\lambda_2^5$ have a local weak left (right) multiplier that is not a weak left (right, respectively) multiplier.

***Keywords:*** associative algebra; left (right) multiplier; derivation; local derivation; local left (right) multiplier.

## Introduction

Associative algebras are classical algebras widely studied by specialists. Associative algebras turned out to be related to other classical algebras, i. e. Lie algebras and Jordan algebras. The classification of finite-dimensional associative algebras is one of the primary areas of modern algebra and, as is known, was first studied by B. Peirce. The classification theorems for 4- and 5-dimensional associative algebras were proved by P. Gabriel [1] and G. Mazzola [2]. The case of associative algebras of dimension ≤4 was discussed by O. Hazlett [3]. The most classification problems for finite-dimensional associative algebras are studied to establish some properties of associative algebras, while the complete classification of associative algebras is still an open problem.

The present article is devoted to weak left and right multipliers and their characterisation. Every derivation is a sum of one weak left and one weak right multipliers. The results of the present paper allow us to study derivation, in particular, to compute the dimension of the space of derivations of an associative algebra. The dimension of the space of derivations of an algebra is an important invariant in the geometric classification of algebras and it has many applications in a number of scientific areas.

In this article, we also study local weak left and right multipliers of finite-dimensional associative algebras introduced in the work [4]. The notion of a weak left (right) multiplier is closely related to the notion of a local derivation. The concept of local derivations goes back to the Gleason – Kahane – Zelazko theorem, which is a fundamental contribution to the theory of Banach algebras. This theorem states that every unital linear functional $F$ on a complex unital Banach algebra $A$ such that $F(a)$ belongs to the spectrum $\sigma(a)$ of the element $a$ for every $a \in A$, is multiplicative (cf. [5; 6]). In view of the modern terminology, this is equivalent to the following condition: any unital linear local homomorphism from an unital complex Banach algebra $A$ to the field $C$ of complex numbers is multiplicative. Recall that a linear mapping $T$ from a Banach algebra $A$ to a Banach algebra $B$ is called a local homomorphism, if for every $a$ in $A$ there exists a homomorphism $\Phi_a : A \to B$, depending on $a$ such that $T(a) = \Phi_a(a)$.

A similar notion was introduced and studied to characterise derivations on operator algebras. Namely, the notion of local derivations was introduced by R. Kadison [7] and D. Larson, A. Sourour [8] independently of each other in 1990. Recall that a linear mapping $\nabla$ of an algebra $A$ into itself is a local derivation, if for each $a$ in $A$, there exists a derivation $D_a$ on $A$ such that $D_a(a) = \nabla(a)$. R. Kadison proves that any continuous local derivation of a von Neumann algebra into its dual Banach bimodule is a derivation. B. Johnson [9] generalises the above result to prove that every local derivation of a $C^*$-algebra into its Banach bimodule is a derivation. Based on these achievements, many authors have studied the local derivations on operator algebras.

The work is structured as follows. In the introduction, we substantiate the problem discussed in this article, present some previously obtained results and bring some basic notions necessary for explaining the results of this paper. Further, in section «Materials and methods» we propose an algorithm for obtaining a common form of the matrix of weak left multipliers on five-dimensional associative algebras $\lambda_1^5$ and $\lambda_2^5$ constructed in proposition 4.3 of the work [10]. Then, in section «Results and discussion», we develop an algorithm for obtaining a common form of the matrix of local weak left multipliers on 5-dimensional associative algebras $\lambda_1^5$ and $\lambda_2^5$. By the results, we conclude that the common form of the matrix of a weak left (right) multiplier on the associative algebras $\lambda_1^5$ and $\lambda_2^5$ does not coincide with the common form of the matrix of a local weak left (right, respectively) multiplier on these algebras. Therefore, the associative algebras $\lambda_1^5$ and $\lambda_2^5$ have a local weak left (right) multiplier that is not a weak left (right, respectively) multiplier.

## Materials and methods

Let $A$ be an associative algebra. Let $\varphi : A \to A$ be a linear mapping. If, for any $x, y \in A$, $\varphi(xy) = \varphi(x)y$ ($\varphi(yx) = y\varphi(x)$), then $\varphi$ is called a weak left (right, respectively) multiplier. Let $L : A \to A$ be a mapping. If there exists $a \in A$ such that $L(x) = ax$ for any $x \in A$, then $L$ is a weak left multiplier, i. e. every left multiplier is a weak left multiplier. Such weak left multiplier we denote as $L_a$.

**Theorem 1** [4]. *Let $A$ be a unital associative algebra, and let $\varphi : A \to A$ be a weak left multiplier. Then $\varphi$ is a left multiplier.*

**Example 1.** Let $K(H)$ be the $C^*$-algebra of all compact operators on a separable infinite-dimensional Hilbert space $H$. For $a \in B(H)$, put $L_a(x) = ax(x \in K(H))$. Then $L_a$ is a weak left multiplier on $K(H)$. But it is not a left multiplier if $a$ does not belong to $K(H)$.

Similar to a weak left multiplier we get the appropriate statements for a weak right multiplier. Let $R : A \to A$ be a mapping. If there exists $a \in A$ such that $R(x) = xa$ for any $x \in A$, then $R$ is a weak right multiplier, i. e. every right multiplier is a weak right multiplier. Such weak right multiplier we denote as $R_a$.

It is clear that, every derivation $D : A \to A$ on an associative algebra $A$ is a subtraction of the weak left multiplier $D(x)y$ and the weak right multiplier $yD(x)$. Similarly, every inner derivation $D_a : A \to A$ on an associative algebra $A$ is a subtraction of the left multiplier $L_a$ and the right multiplier $R_a$. In this case $L_a(e) = R_a(e)$ for the identity element $e \in A$. The following theorem is valid.

Recall that a linear map $D : A \to A$ is called an inner derivation, if there exists $a \in A$ such that $D(x) = ax - xa$ for any element $x \in A$.

**Theorem 2** [4]. *Let $A$ be a unital associative algebra, and let $D : A \to A$ be a derivation. Then $D$ is an inner derivation if and only if there exist a weak left multiplier $\varphi$ and a weak right multiplication $\psi$ such that $D = \varphi - \psi$, $\varphi(e) = \psi(e)$.*

In the present article, a pure algebraic approach to the investigation of multiplier operators and local multipliers on associative algebras is developed. For this propose we use a notion of local left multiplier on an associative algebra as follows: given an associative algebra $A$, a linear map $\Delta : A \to A$ is called a local left multiplier, if for every $x \in A$ there exists an element $a$ in $A$ depending on $x$ such that $\Delta(x) = ax$.

**Theorem 3** [4]. *Let $A$ be an associative division algebra, and let $\psi$ be a local left multiplier on $M_n(A)$. Then $\psi$ is a left multiplier.*

A linear map $\nabla : A \to A$ is called a local inner derivation, if for any element $x \in A$ there exists an element $a \in A$ depending on $x$ such that $\nabla(x) = ax - xa$.

**Theorem 4** [4]. *Let $A$ be a unital division associative algebra, and let $\Delta : M_n(A) \to M_n(A)$ be a local inner derivation on $M_n(A)$. Suppose that there exists a local left multiplier $\varphi$ and a local right multiplier $\psi$ such that $\Delta(x) = \varphi(x) - \psi(x)$, $x \in M_n(A)$. Then $\Delta$ is an inner derivation on $M_n(A)$.*

**Proposition 5** [10]. *Let $A$ be a 5-dimensional naturally graded 2-filiform non-split associative algebras of type $\mu(1, 2)$ over an algebraically closed field $F$ of characteristic zero. Then $A$ is isomorphic to one of the following pairwise non-isomorphic algebras, where the omitted products vanish:*

$$\lambda_1^5 : \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_4e_1 = e_5, \end{cases} \quad \lambda_2^5 : \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_4e_1 = e_5, \\ e_4e_2 = e_5e_1 = e_3. \end{cases}$$

## Results and discussion

**Description of weak left multipliers of the algebras $\lambda_1^5$, $\lambda_2^5$.** Let's consider the following theorem.

**Theorem 6.** *A linear operator on the associative algebra $\lambda_1^5$ is a weak left multiplier if and only if the matrix of this linear operator has the following matrix form:*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ a_{2,1} & a_{1,1} & 0 & a_{2,4} & 0 \\ a_{3,1} & a_{2,1} & a_{1,1} & a_{3,4} & a_{2,4} \\ a_{4,1} & 0 & 0 & a_{4,4} & 0 \\ a_{5,1} & a_{4,1} & 0 & a_{5,4} & a_{4,4} \end{pmatrix}. \qquad (1)$$

P r o o f. Let $a = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_5 e_5$ be an element of $\lambda_1^5$. Then we write $\overline{a} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_5 \end{pmatrix}$.

Let $L : \lambda_1^5 \to \lambda_1^5$ be a weak left multiplier on $\lambda_1^5$, i. e. $L$ is linear and $L(xy) = L(x)y$ for any $x, y \in \lambda_1^5$, and $M = (a_{ij})_{ij=1}^{5}$ be the matrix of $L$. Then

$$\overline{L(a)} = M\overline{a}.$$

We compute the components of the matrix $M$. By the multiplication table for $\lambda_1^5$ and the definition of a weak left multiplier, we have:

1) from $L(e_1e_1) = L(e_1)e_1 = L(e_2)$ it follows that $a_{1,2}e_1 + a_{2,2}e_2 + a_{3,2}e_3 + a_{4,2}e_4 + a_{5,2}e_5 = a_{1,1}e_2 + a_{2,1}e_3 + a_{4,1}e_5$, and, hence, $a_{1,2} = 0$, $a_{1,1} = a_{2,2}$, $a_{2,1} = a_{3,2}$, $a_{4,2} = 0$, $a_{4,1} = a_{5,2}$;

2) from $L(e_1e_2) = L(e_1)e_2 = L(e_3)$ it follows that $a_{1,3}e_1 + a_{2,3}e_2 + a_{3,3}e_3 + a_{4,3}e_4 + a_{5,3}e_5 = a_{1,1}e_3$, and, hence, $a_{1,3} = 0$, $a_{2,3} = 0$, $a_{4,3} = 0$, $a_{5,3} = 0$, $a_{1,1} = a_{3,3}$;

3) from $L(e_2e_1) = L(e_2)e_1 = L(e_3)$ it follows that $a_{1,3}e_1 + a_{2,3}e_2 + a_{3,3}e_3 + a_{4,3}e_4 + a_{5,3}e_5 = a_{1,2}e_2 + a_{2,2}e_3 + a_{4,2}e_5$, and, hence, $a_{1,3} = 0$, $a_{2,3} = a_{1,2}$, $a_{3,3} = a_{2,2}$, $a_{4,3} = 0$, $a_{4,2} = a_{5,3} = 0$;

4) from $L(e_2e_2) = L(e_2)e_2 = a_{1,2}e_3 = 0$ it follows that $a_{1,2} = 0$;

5) from $L(e_3e_1) = L(e_3)e_1$ it follows that $a_{1,3}e_2 + a_{2,3}e_3 + a_{4,3}e_5 = 0$, and, hence, $a_{1,3} = 0$, $a_{2,3} = 0$, $a_{4,3} = 0$;

6) from $L(e_3e_2) = L(e_3)e_2 = 0$ it follows that $a_{1,3}e_3 = 0$, and, hence, $a_{1,3} = 0$;

7) from $L(e_4e_1) = L(e_4)e_1 = L(e_5)$ it follows that $a_{1,5}e_1 + a_{2,5}e_2 + a_{3,5}e_3 + a_{4,5}e_4 + a_{5,5}e_5 = a_{1,4}e_2 + a_{2,4}e_3 + a_{4,4}e_5$, and, hence, $a_{1,5} = 0$, $a_{1,4} = a_{2,5}$, $a_{2,4} = a_{3,5}$, $a_{4,5} = 0$, $a_{4,4} = a_{5,5}$;

8) from $L(e_4e_2) = L(e_4)e_2$ it follows that $a_{1,4}e_3 = 0$, and, hence, $a_{1,4} = 0$;

9) from $L(e_5e_1) = L(e_5)e_1 = 0$ it follows that $a_{1,5}e_2 + a_{2,5}e_3 + a_{4,5}e_5 = 0$, and, hence, $a_{4,5} = 0$, $a_{1,5} = 0$, $a_{2,5} = 0$, $a_{1,4} = 0$;

10) from $L(e_5e_2) = L(e_5)e_2 = 0$ it follows that $a_{1,5}e_3 = 0$, and, hence, $a_{1,5} = 0$.

As the result we get the following matrix:

$$M = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ a_{2,1} & a_{1,1} & 0 & a_{2,4} & 0 \\ a_{3,1} & a_{2,1} & a_{1,1} & a_{3,4} & a_{2,4} \\ a_{4,1} & 0 & 0 & a_{4,4} & 0 \\ a_{5,1} & a_{4,1} & 0 & a_{5,4} & a_{4,4} \end{pmatrix}.$$

This matrix coincides with matrix (1).

Now we prove that the matrix $M$ defines a weak left multiplier on $\lambda_1^5$. Let $a = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_5 e_5$, $b = \beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_5 e_5$ be an element of $\lambda_1^5$. Then

$$\overline{L(ab)} = M\overline{ab} = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 & 0 \\ a_{2,1} & a_{1,1} & 0 & a_{2,4} & 0 \\ a_{3,1} & a_{2,1} & a_{1,1} & a_{3,4} & a_{2,4} \\ a_{4,1} & 0 & 0 & a_{4,4} & 0 \\ a_{5,1} & a_{4,1} & 0 & a_{5,4} & a_{4,4} \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_1\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \\ 0 \\ \alpha_4\beta_1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ a_{1,1}\alpha_1\beta_1 \\ a_{2,1}\alpha_1\beta_1 + a_{1,1}(\alpha_1\beta_2 + \alpha_2\beta_1) + a_{2,4}\alpha_4\beta_1 \\ 0 \\ a_{4,4}\alpha_4\beta_1 + a_{4,1}\alpha_1\beta_1 \end{pmatrix}.$$

At the same time,

$$\overline{L(a)} = \begin{pmatrix} a_{1,1}\alpha_1 \\ \alpha_{2,1}\alpha_1 + a_{1,1}\alpha_2 + a_{2,4}\alpha_4 \\ \alpha_{3,1}\alpha_1 + a_{2,1}\alpha_2 + a_{1,1}\alpha_3 + a_{3,4}\alpha_4 + a_{2,4}\alpha_5 \\ a_{4,1}\alpha_1 + a_{4,4}\alpha_4 \\ a_{5,1}\alpha_1 + a_{4,1}\alpha_2 + a_{5,4}\alpha_4 + a_{4,4}\alpha_5 \end{pmatrix},$$

$$\overline{L(a)b} = \begin{pmatrix} 0 \\ a_{1,1}\alpha_1\beta_1 \\ a_{2,1}\alpha_1\beta_1 + a_{1,1}(\alpha_1\beta_2 + \alpha_2\beta_1) + a_{2,4}\alpha_4\beta_1 \\ 0 \\ a_{4,4}\alpha_4\beta_1 + a_{4,1}\alpha_1\beta_1 \end{pmatrix}.$$

From this it follows that $L(ab) = L(a)b$ for any elements $a$ and $b$ in $\lambda_1^5$. Therefore, this common form of a matrix is sufficient to the linear operator on $\lambda_1^5$, generated by the matrix $M$, be a weak left multiplier on $\lambda_1^5$. This ends the proof.

The following theorem is proven similar to the proof of theorem 6.

**Theorem 7.** *A linear operator on the associative algebra* $\lambda_2^5$ *is a weak left multiplier if and only if the matrix of this linear operator has the following matrix form:*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & a_{1,4} & 0 \\ a_{2,1} & a_{1,1} & 0 & a_{2,4} & a_{1,4} \\ a_{3,1} & a_{2,1} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{3,3} - a_{1,1} & 0 & 0 & a_{3,3} - a_{1,4} & 0 \\ 0 & a_{3,3} - a_{1,1} & 0 & a_{3,5} - a_{2,4} & a_{3,3} - a_{1,4} \end{pmatrix}. \qquad (2)$$

**Description of local weak left multipliers of the algebras** $\lambda_1^5$, $\lambda_2^5$. In the present article, a pure algebraic approach to the investigation of multiplier operators and local multipliers on associative algebras is developed. For this propose we use a notion of a local weak left multiplier on an associative algebra as follows: given an associative algebra $A$, a linear map $\Delta : A \to A$ is called a local weak left multiplier, if for every $x \in A$ there exists a weak left multiplier $L$ on $A$, depending on $x$ such that $\Delta(x) = L(x)$.

**Theorem 8.** *A linear operator on the associative algebra* $\lambda_1^5$ *is a local weak left multiplier if and only if the matrix of this linear operator has the following matrix form:*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & b_{2,4} & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & 0 & 0 & b_{4,4} & 0 \\ b_{5,1} & b_{5,2} & 0 & b_{5,4} & b_{5,5} \end{pmatrix}. \qquad (3)$$

P r o o f. Let $\nabla$ be a local weak left multiplier on $\lambda_1^5$, let $B = \left(b_{i,j}\right)_{i,j=1}^5$ be the matrix, defining the linear operator $\nabla$. By the definition, for any element $x \in \lambda_1^5$ there exists a weak left multiplier operator $L_x$ such that $\nabla(x) = L_x(x)$. Then, for the appropriate matrix $A_x$ of the operator $L_x$ we have

$$\overline{L_x(x)} = B\overline{x} = A_x\overline{x},\ A_x = \left(a_{i,j}^x\right)_{i,j=1}^5. \tag{4}$$

By form (1) of the matrix of a weak left multiplier on $\lambda_1^5$, using equalities $\nabla(e_i) = L_{e_i}(e_i) = B\overline{e}_i$, $i = 1, 2, 3, 4, 5$, we get

$$b_{1,1} = a_{1,1}^{e_1},\ b_{2,1} = a_{2,1}^{e_1},\ b_{3,1} = a_{3,1}^{e_1},\ b_{4,1} = a_{4,1}^{e_1},\ b_{5,1} = a_{5,1}^{e_1},$$
$$b_{1,2} = 0,\ b_{2,2} = a_{1,1}^{e_2},\ b_{3,2} = a_{2,1}^{e_2},\ b_{4,2} = 0,\ b_{5,2} = a_{4,1}^{e_2},$$
$$b_{1,3} = 0,\ b_{2,3} = 0,\ b_{3,3} = a_{1,1}^{e_3},\ b_{4,3} = 0,\ b_{5,3} = 0,$$
$$b_{1,4} = 0,\ b_{2,4} = a_{2,4}^{e_4},\ b_{3,4} = a_{3,4}^{e_4},\ b_{4,4} = a_{4,4}^{e_4},\ b_{5,4} = a_{5,4}^{e_4},$$
$$b_{1,5} = 0,\ b_{2,5} = 0,\ b_{3,5} = a_{2,4}^{e_5},\ b_{4,5} = 0,\ b_{5,5} = a_{4,4}^{e_5}.$$

Thus, according to (4) we get the following system of linear equations:

$$\begin{cases} x_1a_{1,1}^x = x_1a_{1,1}^{e_1}, \\ x_1a_{2,1}^x + x_2a_{1,1}^x + x_4a_{2,4}^x = x_1a_{2,1}^{e_1} + x_2a_{1,1}^{e_2} + x_4a_{2,4}^{e_4}, \\ x_1a_{3,1}^x + x_2a_{2,1}^x + x_3a_{1,1}^x + x_4a_{3,4}^x + x_5a_{2,4}^x = x_1a_{3,1}^{e_1} + x_2a_{2,1}^{e_2} + x_3a_{1,1}^{e_3} + x_4a_{3,4}^{e_4} + x_5a_{2,4}^{e_5}, \\ x_1a_{4,1}^x + x_4a_{4,4}^x = x_1a_{4,1}^{e_1} + x_4a_{4,4}^{e_4}, \\ x_1a_{5,1}^x + x_2a_{4,1}^x + x_4a_{5,4}^x + x_5a_{4,4}^x = x_1a_{5,1}^{e_1} + x_2a_{4,1}^{e_2} + x_4a_{5,4}^{e_4} + x_5a_{4,4}^{e_5}. \end{cases} \tag{5}$$

If for each element $x \in \lambda_1^5$ there exists a matrix $A_x$ of form (1) such that

$$B\overline{x} = A_x\overline{x},$$

then the linear operator, defined by the matrix $B$ is a local weak left multiplier. In other words, if for each element $x \in \lambda_1^5$ system of linear equations (5) has a solution with respect to the variables

$$a_{1,1}^x,\ a_{2,1}^x,\ a_{2,4}^x,\ a_{3,1}^x,\ a_{3,4}^x,\ a_{4,1}^x,\ a_{4,4}^x,\ a_{5,1}^x,\ a_{5,4}^x,$$

then the linear operator, defined by the matrix $B$ is a local weak left multiplier. Note that, if the left part of any equation of system (5) is equal to zero, then the right part of this equation is also equal to zero. We show that for each element $x \in \lambda_1^5$, system of linear equations (5) has a solution.

Now, suppose that $x_1 \neq 0$. Then, from (5) it follows that

$$\begin{cases} a_{1,1}^x = a_{1,1}^{e_1}, \\ x_1a_{2,1}^x + x_4a_{2,4}^x = x_1a_{2,1}^{e_1} + x_2a_{1,1}^{e_2} + x_4a_{2,4}^{e_4} - x_2a_{1,1}^{e_1}, \\ x_1a_{3,1}^x + x_2a_{2,1}^x + x_4a_{3,4}^x + x_5a_{2,4}^x = x_1a_{3,1}^{e_1} + x_2a_{2,1}^{e_2} + x_3a_{1,1}^{e_3} + x_4a_{3,4}^{e_4} + x_5a_{2,4}^{e_5} - x_3a_{1,1}^{e_1}, \\ x_1a_{4,1}^x + x_4a_{4,4}^x = x_1a_{4,1}^{e_1} + x_4a_{4,4}^{e_4}, \\ x_1a_{5,1}^x + x_2a_{4,1}^x + x_4a_{5,4}^x + x_5a_{4,4}^x = x_1a_{5,1}^{e_1} + x_2a_{4,1}^{e_2} + x_4a_{5,4}^{e_4} + x_5a_{4,4}^{e_5}. \end{cases}$$

It is not hard to see that the last system of linear equations has solution for each $x_1 \neq 0$ from the field $F$.

Now, suppose that $x_1 = 0$, $x_4 \neq 0$. Then we have

$$\begin{cases} 0 = 0, \\ x_2a_{1,1}^x + x_4a_{2,4}^x = x_2a_{1,1}^{e_2} + x_4a_{2,4}^{e_4}, \\ x_2a_{2,1}^x + x_3a_{1,1}^x + x_4a_{3,4}^x + x_5a_{2,4}^x = x_2a_{2,1}^{e_2} + x_3a_{1,1}^{e_3} + x_4a_{3,4}^{e_4} + x_5a_{2,4}^{e_5}, \\ x_4a_{4,4}^x = x_4a_{4,4}^{e_4}, \\ x_2a_{4,1}^x + x_4a_{5,4}^x + x_5a_{4,4}^x = x_2a_{4,1}^{e_2} + x_4a_{5,4}^{e_4} + x_5a_{4,4}^{e_5}. \end{cases}$$

By virtue of distinct variables $a_{2,4}^x$, $a_{3,4}^x$, $a_{4,4}^x$ and $a_{5,4}^x$ with the nonzero coefficient $x_4$ this system of linear equations always has a solution. If $x_1 = 0$, $x_4 = 0$, then we have the following system of linear equations:

$$\begin{cases} 0 = 0, \\ x_2 a_{1,1}^{x} = x_2 a_{1,1}^{e_2}, \\ x_2 a_{2,1}^{x} + x_3 a_{1,1}^{x} + x_5 a_{2,4}^{x} = x_2 a_{2,1}^{e_2} + x_3 a_{1,1}^{e_3} + x_5 a_{2,4}^{e_5}, \\ 0 = 0, \\ x_2 a_{4,1}^{x} + x_5 a_{4,4}^{x} = x_2 a_{4,1}^{e_2} + x_5 a_{4,4}^{e_5}. \end{cases}$$

It is clear that, if $x_2 \neq 0$, then the last system of linear equations always has a solution.

Now, suppose that $x_1 = 0$, $x_4 = 0$, $x_2 = 0$. Then

$$\begin{cases} 0 = 0, \\ 0 = 0, \\ x_3 a_{1,1}^{x} + x_5 a_{2,4}^{x} = x_3 a_{1,1}^{e_3} + x_5 a_{2,4}^{e_5}, \\ 0 = 0, \\ x_5 a_{4,4}^{x} = x_5 a_{4,4}^{e_5}. \end{cases}$$

It is clear that, for any $x_3$ and $x_5$ the last system of linear equations has a solution. Thus, system of linear equations (5) always has a solution, i. e. the linear operator, generated by the matrix

$$B = \begin{pmatrix} a_{1,1}^{e_1} & 0 & 0 & 0 & 0 \\ a_{2,1}^{e_1} & a_{1,1}^{e_2} & 0 & a_{2,4}^{e_4} & 0 \\ a_{3,1}^{e_1} & a_{2,1}^{e_2} & a_{1,1}^{e_3} & a_{3,4}^{e_4} & a_{2,4}^{e_5} \\ a_{4,1}^{e_1} & 0 & 0 & a_{4,4}^{e_4} & 0 \\ a_{5,1}^{e_1} & a_{4,1}^{e_2} & 0 & a_{5,4}^{e_4} & a_{4,4}^{e_5} \end{pmatrix}$$

is a local weak left multiplier. Since the local weak left multiplier was chosen arbitrarily we can the matrix $B$ rewrite in the following form:

$$B = \begin{pmatrix} b_{1,1} & 0 & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & b_{2,4} & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & 0 & 0 & b_{4,4} & 0 \\ b_{5,1} & b_{5,2} & 0 & b_{5,4} & b_{5,5} \end{pmatrix}.$$

This matrix coincides with matrix (3). The proof is completed.

**Theorem 9.** *A linear operator on the associative algebra $\lambda_2^5$ is a weak left multiplier if and only if the matrix of this linear operator has the following matrix form:*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & b_{1,4} & 0 \\ b_{2,1} & b_{2,2} & 0 & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & 0 & 0 & b_{4,4} & 0 \\ b_{5,1} & b_{5,2} & 0 & b_{5,4} & b_{5,5} \end{pmatrix}. \tag{6}$$

Proof. Let $\nabla$ be a local weak left multiplier on $\lambda_2^5$, let $B = \left(b_{ij}\right)_{ij=1}^{5}$ be the matrix, generating the linear operator $\nabla$. By the definition, for any element $x \in \lambda_2^5$ there exists a weak left multiplier operator $L_x$ such that $\nabla(x) = L_x(x)$. Then for the appropriate matrix $A_x$ of the operator $L_x$, we have

$$\overline{L_x(x)} = B\overline{x} = A_x\overline{x},\ A_x = \left(a_{i,\,j}^{x}\right)_{i,\,j=1}^{5}. \tag{7}$$

By form (2) of the matrix of a weak left multiplier on $\lambda_2^5$, using equalities $\nabla(e_i) = L_{e_i}(e_i) = B\overline{e}_i$, $i = 1, 2, 3, 4, 5$, we get

$$b_{1,1} = a_{1,1}^{e_1},\ b_{2,1} = a_{2,1}^{e_1},\ b_{3,1} = a_{3,1}^{e_1},\ b_{4,1} = a_{3,3}^{e_1} - a_{1,1}^{e_1},\ b_{5,1} = 0,$$
$$b_{1,2} = 0,\ b_{2,2} = a_{1,1}^{e_2},\ b_{3,2} = a_{2,1}^{e_2},\ b_{4,2} = 0,\ b_{5,2} = a_{3,3}^{e_2} - a_{1,1}^{e_2},$$
$$b_{1,3} = 0,\ b_{2,3} = 0,\ b_{3,3} = a_{3,3}^{e_3},\ b_{4,3} = 0,\ b_{5,3} = 0,$$
$$b_{1,4} = a_{4,1}^{e_4},\ b_{2,4} = a_{2,4}^{e_4},\ b_{3,4} = a_{3,4}^{e_4},\ b_{4,4} = a_{3,3}^{e_4} - a_{1,4}^{e_4},\ b_{5,4} = a_{3,5}^{e_4} - a_{2,4}^{e_4},$$
$$b_{1,5} = 0,\ b_{2,5} = a_{4,1}^{e_5},\ b_{3,5} = a_{3,5}^{e_5},\ b_{4,5} = 0,\ b_{5,5} = a_{3,3}^{e_5} - a_{1,4}^{e_5}.$$

Thus, according to (7) we get the following system of linear equations:

$$\begin{cases} x_1 a_{1,1}^x + x_4 a_{1,4}^x = x_1 a_{1,1}^{e_1} + x_4 a_{1,4}^{e_4}, \\ x_1 a_{2,1}^x + x_2 a_{1,1}^x + x_4 a_{2,4}^x + x_5 a_{1,4}^x = x_1 a_{2,1}^{e_1} + x_2 a_{1,1}^{e_2} + x_4 a_{2,4}^{e_4} + x_5 a_{1,4}^{e_5}, \\ x_1 a_{3,1}^x + x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_4 a_{3,4}^x + x_5 a_{3,5}^x = x_1 a_{3,1}^{e_1} + x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_4 a_{3,4}^{e_4} + x_5 a_{3,5}^{e_5}, \\ x_1\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,3}^x - a_{1,4}^x\right) = x_1\left(a_{3,3}^{e_1} - a_{1,1}^{e_1}\right) + x_4\left(a_{3,3}^{e_4} - a_{1,4}^{e_4}\right), \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^{e_2} - a_{1,1}^{e_2}\right) + x_4\left(a_{3,5}^{e_4} - a_{2,4}^{e_4}\right) + x_5\left(a_{3,3}^{e_5} - a_{1,4}^{e_5}\right). \end{cases} \quad (8)$$

If for each element $x \in \lambda_1^5$ there exists a matrix $A_x$ of form (2) such that

$$B\overline{x} = A_x \overline{x},$$

then the linear operator, defined by the matrix $B$ is a local weak left multiplier. In other words, if for each element $x \in \lambda_1^5$ system of linear equations (8) has a solution with respect to the variables

$$a_{1,1}^x,\ a_{2,1}^x,\ a_{2,4}^x,\ a_{3,1}^x,\ a_{3,4}^x,\ a_{4,1}^x,\ a_{4,4}^x,\ a_{5,1}^x,\ a_{5,4}^x, \quad (9)$$

then the linear operator, defined by the matrix $B$ is a local weak left multiplier. Note that, if the left part of any equation of system (8) is equal to zero, then the right part of this equation is also equal to zero. We show that for each element $x \in \lambda_1^5$ system of linear equations (8) has a solution.

Now, suppose that $x_1 \neq 0$. System of linear equations (8) is equivalent to the following system of linear equations:

$$\begin{cases} x_1 a_{1,1}^x + x_4 a_{1,4}^x = x_1 a_{1,1}^{e_1} + x_4 a_{1,4}^{e_4}, \\ x_1 a_{2,1}^x + x_2 a_{1,1}^x + x_4 a_{2,4}^x + x_5 a_{1,4}^x = x_1 a_{2,1}^{e_1} + x_2 a_{1,1}^{e_2} + x_4 a_{2,4}^{e_4} + x_5 a_{1,4}^{e_5}, \\ x_1 a_{3,1}^x + x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_4 a_{3,4}^x + x_5 a_{3,5}^x = x_1 a_{3,1}^{e_1} + x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_4 a_{3,4}^{e_4} + x_5 a_{3,5}^{e_5}, \\ \left(x_1 + x_4\right) a_{3,3}^x = x_1 a_{3,3}^{e_1} + x_4 a_{3,3}^{e_4}, \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right). \end{cases} \quad (10)$$

The order of computation of the values of variables (9) is the following:

$$\begin{cases} x_1 a_{1,1}^x + x_4 a_{1,4}^x = x_1 a_{1,1}^{e_1} + x_4 a_{1,4}^{e_4}, \\ x_1 a_{2,1}^x + x_2 a_{1,1}^x + x_4 a_{2,4}^x + x_5 a_{1,4}^x = x_1 a_{2,1}^{e_1} + x_2 a_{1,1}^{e_2} + x_4 a_{2,4}^{e_4} + x_5 a_{1,4}^{e_5}, \\ \left(x_1 + x_4\right) a_{3,3}^x = x_1 a_{3,3}^{e_1} + x_4 a_{3,3}^{e_4}, \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^{e_2} - a_{1,1}^{e_2}\right) + x_4\left(a_{3,5}^{e_4} - a_{2,4}^{e_4}\right) + x_5\left(a_{3,3}^{e_5} - a_{1,4}^{e_5}\right), \\ x_1 a_{3,1}^x + x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_4 a_{3,4}^x + x_5 a_{3,5}^x = x_1 a_{3,1}^{e_1} + x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_4 a_{3,4}^{e_4} + x_5 a_{3,5}^{e_5}. \end{cases}$$

Note that the variables $a_{1,1}^x,\ a_{2,1}^x,\ a_{3,3}^x,\ a_{3,1}^x$ have a nonzero coefficient, if $x_1 + x_4 \neq 0$ and system of linear equations (10) has a solution. If $x_1 \neq 0$ and $x_1 + x_4 = 0$, then $x_4 = -x_1$ and similar to the previous case system of linear equations (10) has a solution. Thus, system of linear equations (8) has a solution for each $x_1 \neq 0$ from the field $F$.

Now, suppose that $x_1 = 0$, $x_4 \neq 0$, then system of linear equations (8) is equivalent to the following system of linear equations:

$$\begin{cases} x_4 a_{1,4}^x = x_4 a_{1,4}^{e_4}, \\ x_2 a_{1,1}^x + x_4 a_{2,4}^x + x_5 a_{1,4}^x = x_2 a_{1,1}^{e_2} + x_4 a_{2,4}^{e_4} + x_5 a_{1,4}^{e_5}, \\ x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_4 a_{3,4}^x + x_5 a_{3,5}^x = x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_4 a_{3,4}^{e_4} + x_5 a_{3,5}^{e_5}, \\ x_4 a_{3,3}^x = x_4 a_{3,3}^{e_4}, \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^{e_2} - a_{1,1}^{e_2}\right) + x_4\left(a_{3,5}^{e_4} - a_{2,4}^{e_4}\right) + x_5\left(a_{3,3}^{e_5} - a_{1,4}^{e_5}\right). \end{cases}$$

In this case the variables $a_{1,4}^x$, $a_{2,4}^x$, $a_{3,4}^x$, $a_{3,3}^x$, $a_{3,5}^x$ have a nonzero coefficient, and the order of computation of the values of these variables is the following:

$$\begin{cases} x_4 a_{1,4}^x = x_4 a_{1,4}^{e_4}, \\ x_2 a_{1,1}^x + x_4 a_{2,4}^x + x_5 a_{1,4}^x = x_2 a_{1,1}^{e_2} + x_4 a_{2,4}^{e_4} + x_5 a_{1,4}^{e_5}, \\ x_4 a_{3,3}^x = x_4 a_{3,3}^{e_4}, \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_4\left(a_{3,5}^x - a_{2,4}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^{e_2} - a_{1,1}^{e_2}\right) + x_4\left(a_{3,5}^{e_4} - a_{2,4}^{e_4}\right) + x_5\left(a_{3,3}^{e_5} - a_{1,4}^{e_5}\right), \\ x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_4 a_{3,4}^x + x_5 a_{3,5}^x = x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_4 a_{3,4}^{e_4} + x_5 a_{3,5}^{e_5}. \end{cases}$$

As the result we have a solution of (10). Thus, system of linear equations (8) has a solution in the case of $x_1 = 0$, $x_4 \neq 0$.

Now, suppose that $x_1 = 0$, $x_4 = 0$. Then system of linear equations (8) is equivalent to the following system of linear equations:

$$\begin{cases} 0 = 0, \\ x_2 a_{1,1}^x + x_5 a_{1,4}^x = x_2 a_{1,1}^{e_2} + x_5 a_{1,4}^{e_5}, \\ x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_5 a_{3,5}^x = x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_5 a_{3,5}^{e_5}, \\ 0 = 0, \\ x_2\left(a_{3,3}^x - a_{1,1}^x\right) + x_5\left(a_{3,3}^x - a_{1,4}^x\right) = x_2\left(a_{3,3}^{e_2} - a_{1,1}^{e_2}\right) + x_5\left(a_{3,3}^{e_5} - a_{1,4}^{e_5}\right), \end{cases}$$

i. e.

$$\begin{cases} 0 = 0, \\ x_2 a_{1,1}^x + x_5 a_{1,4}^x = x_2 a_{1,1}^{e_2} + x_5 a_{1,4}^{e_5}, \\ x_2 a_{2,1}^x + x_3 a_{3,3}^x + x_5 a_{3,5}^x = x_2 a_{2,1}^{e_2} + x_3 a_{3,3}^{e_3} + x_5 a_{3,5}^{e_5}, \\ 0 = 0, \\ \left(x_2 + x_5\right) a_{3,3}^x = x_2 a_{3,3}^{e_2} + x_5 a_{3,3}^{e_5}. \end{cases}$$

Similar to the previous cases this system of linear equations also has a solution. Thus, system of linear equations (8) always has a solution, i. e. the linear operator, generated by the matrix

$$B = \begin{pmatrix} a_{1,1}^{e_1} & 0 & 0 & a_{1,4}^{e_4} & 0 \\ a_{2,1}^{e_1} & a_{1,1}^{e_2} & 0 & a_{2,4}^{e_4} & a_{1,4}^{e_5} \\ a_{3,1}^{e_1} & a_{2,1}^{e_2} & a_{3,3}^{e_3} & a_{3,4}^{e_4} & a_{3,5}^{e_5} \\ a_{3,3}^{e_1} - a_{1,1}^{e_1} & 0 & 0 & a_{3,3}^{e_4} - a_{1,4}^{e_4} & 0 \\ 0 & a_{3,3}^{e_2} - a_{1,1}^{e_2} & 0 & a_{3,5}^{e_4} - a_{2,4}^{e_4} & a_{3,3}^{e_5} - a_{1,4}^{e_5} \end{pmatrix},$$

is a local weak left multiplier. Since the local weak left multiplier was chosen arbitrarily we can rewrite the matrix $B$ in the following form:

$$B=\begin{pmatrix} b_{1,1} & 0 & 0 & b_{1,4} & 0 \\ b_{2,1} & b_{2,2} & 0 & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & 0 & 0 & b_{4,4} & 0 \\ 0 & b_{5,2} & 0 & b_{5,4} & b_{5,5} \end{pmatrix}.$$

This matrix coincides with matrix (6). This completes the proof.

## Conclusions

Note that the common form of the matrix of a local weak left (right) multiplier on an algebra includes the common form of the matrix of a weak left (right, respectively) multiplier on this algebra. The coincidence of these common forms denotes that every local weak left (right) multiplier of the considering algebra is a weak left (right, respectively) multiplier. But the common form of the matrix of a weak left (right) multiplier on the associative algebras $\lambda_1^5$ and $\lambda_2^5$ does not coincide with the common form of the matrix of a local weak left (right, respectively) multiplier on these algebras by theorems 6, 8 and 7, 9, respectively. Therefore, the associative algebras $\lambda_1^5$ and $\lambda_2^5$ have local weak left (right) multipliers that are not weak left (right, respectively) multipliers.

We note that local weak left (right) multipliers of an arbitrary low-dimension algebra can be similarly described using a common form of the matrix of weak left (right, respectively) multipliers on this algebra. A technique for constructing a local weak left (right) multiplier, which is not a weak left (right, respectively) multiplier, developed by us, can be applied to an arbitrary low-dimension algebra, weak left (right) multipliers of which have a matrix of a generalised form.

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