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ДИФФЕРЕНЦИРОВАНИЯ ПРОСТЫХ ТРЕХМЕРНЫХ АНТИКОММУТАТИВНЫХ АЛГЕБР

***С. В. ПЧЕЛИНЦЕВ*[1], *М. С. ДУБРОВИН*[2]**

[1]*Финансовый университет при Правительстве Российской Федерации, пр. Ленинградский, 49/2, 125167, г. Москва, Россия*
[2]*Московский городской педагогический университет, 2-й Сельскохозяйственный проезд, 4, корп. 1, 129226, г. Москва, Россия*

Аннотация. Исследуются алгебры дифференцирований простых трехмерных антикоммутативных алгебр над алгебраически замкнутым полем. Главное утверждение статьи заключается в том, что алгебры дифференцирований простых трехмерных антикоммутативных алгебр имеют размерности 0, 1 и 3, в последнем случае они изоморфны простой алгебре Ли sl_2 бесследных матриц порядка 2.

Ключевые слова: алгебры дифференцирований; алгебры Ли; простые трехмерные антикоммутативные алгебры.

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Авторы:
Сергей Валентинович Пчелинцев – доктор физико-математических наук; профессор кафедры анализа данных и машинного обучения факультета информационных технологий и анализа больших данных.
Максим Станиславович Дубровин – аспирант кафедры математической логики, алгебры, теории чисел и дискретной математики Института цифрового образования. Научный руководитель – С. В. Пчелинцев.

Authors:
Sergey V. Pchelintsev, doctor of science (physics and mathematics); professor at the department of data analysis and machine learning, faculty of information technology and big data analysis.
pchelinzev@mail.ru
Maxim S. Dubrovin, postgraduate student at the department of mathematical logic, algebra, number theory and discrete mathematics, Institute of Digital Education.
m.s.dubrovin@yandex.ru

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DERIVATIONS OF SIMPLE THREE-DIMENSIONAL ANTICOMMUTATIVE ALGEBRAS

S. V. PCHELINTSEV [a]**,** ***M. S. DUBROVIN*** [b]

[a]*Financial University under the Government of the Russian Federation, 49/2 Leningradskij Avenue, Moscow 125167, Russia*
[b]*Moscow City University, 4 Vtoroj Selskohoziajstvenny proezd, 1 building, Moscow 129226, Russia*

Corresponding author: M. S. Dubrovin (m.s.dubrovin@yandex.ru)

Abstract. In this paper, we investigate the derivation algebras of simple three-dimensional anticommutative algebras over algebraically closed fields. The main statement of the article is that the derivation algebras of simple three-dimensional anticommutative algebras have dimensions 0, 1 and 3, for the latter case they are isomorphic to a simple Lie algebra of traceless matrices of the 2^{nd} order.

Keywords: derivation algebras; Lie algebras; simple three-dimensional anticommutative algebras.

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Introduction

The study of derivation algebras is an important aspect of modern mathematics [1–9]; when discussing algebras, it is assumed that they are considered over an algebraically closed field even if it is not explicitly stated.

It is well known that any derivation of a simple finite-dimensional associative algebra is internal [8]. Similar results are valid for simple finite-dimensional alternative, Jordan and Lie algebras [5–7]. For a simple algebra A from the indicated classes, the Lie algebra $\operatorname{Der}(A)$, which is a derivation algebra for A, is simple.

In particular, the derivation algebra $\operatorname{Der}(A)$ of a simple split three-dimensional Lie algebra A is isomorphic to the algebra A.

In this paper, we study the derivation algebras of simple three-dimensional anticommutative algebras over algebraically closed fields. Additionally, algebraic structures on finite-dimensional algebras, similar to those examined in this paper, are studied, for example, in [9].

Any one-dimensional anticommutative algebra has zero multiplication, thus, it is not simple. Let us first demonstrate that there are no two-dimensional simple anticommutative algebras.

Let A be such an algebra, in particular. Consider $a \in A$ such that its right multiplication operator $R_a : A \to A$, $xR_a = xa$ is non-zero. In some basis (e_1, e_2), this operator is either diagonal or a Jordan block, i. e.,

$$M_{e_1, e_2}(R_a) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \text{ or } M_{e_1, e_2}(R_a) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Examine each of these cases:

1) $M_{e_1, e_2}(R_a) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. One of the scalars α_1, α_2 is non-zero; let $\alpha_1 \neq 0$. Then, due to anticommutativity, the vectors e_1, a are not proportional. Consequently, they form a basis for A, and the linear span $\langle e_1 \rangle$ is a proper ideal in A, contradicting the simplicity of algebra A;

2) $M_{e_1, e_2}(R_a) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. If $\lambda \neq 0$, then $\langle e_2 \rangle$ is a proper ideal in A. If $\lambda = 0$, then $e_1 a = e_2$. Decompose a along the basis (e_1, e_2): $a = \alpha e_1 + \beta e_2$. Due to anticommutativity, $\beta \neq 0$ and $e_2 = e_1 a = e_1(\alpha e_1 + \beta e_2) = \beta e_1 e_2$. Thus, $\langle e_2 \rangle$ is a proper ideal in A.

The main goal of this paper is to prove the following theorem.

Theorem. *Let M be a simple three-dimensional anticommutative algebra over an algebraically closed field of a characteristic different from 2. Then its derivation algebra, denoted as* $\operatorname{Der}(M)$*, has a dimension of 0, 1 or 3. If* $\dim \operatorname{Der}(M) = 3$*, then the algebra* $\operatorname{Der}(M)$ *is isomorphic to the simple three-dimensional Lie algebra* sl_2.

It is known [10] that every simple three-dimensional anticommutative algebra is isotopically simple and isomorphic to one of the algebras with basis (a, b, c):

$$A = A(\lambda, \alpha, \beta, \gamma): ba = \lambda b + c,\ ca = \lambda c,\ bc = \alpha a + \beta b + \gamma c,\ \alpha\lambda \neq 0,$$

$$B = B(\lambda, \mu, \alpha, \beta, \gamma): ba = \lambda b,\ ca = \mu c,\ bc = \alpha a + \beta b + \gamma c,\ \alpha\lambda\mu \neq 0.$$

Lemma. *The algebra* $A = A(\lambda, \alpha, \beta, \gamma)$ *is not a Lie algebra. The algebra* $B = B(\lambda, \mu, \alpha, \beta, \gamma)$ *is a Lie algebra if and only if* $\beta = \gamma = \lambda + \mu = 0$.

P r o o f. Let us calculate the Jacobian in the algebra A for the basis elements:

$$J(a, b, c) = (ab)c + (bc)a + (ca)b = -(\lambda b + c)c + (\alpha a + \beta b + \gamma c)a + (\lambda c)b =$$
$$= -\lambda bc + \beta ba + \gamma ca + (\lambda c)b = -2\lambda bc + \beta ba + \gamma ca.$$

Since in the algebra A, the elements bc, ba, ca are linearly independent, it follows from $J(a, b, c) = 0$ that $\lambda = 0$. However, according to the condition, $\lambda \neq 0$, which means that A is not a Lie algebra.

The second statement is proved in a similar manner:

$$J(a, b, c) = (ab)c + (bc)a + (ca)b = -(\lambda b)c + (\alpha a + \beta b + \gamma c)a + (\mu c)b =$$
$$= -\lambda bc + \beta ba + \gamma ca + \mu cb = \beta ba + \gamma ca + (\lambda + \mu)cb.$$

Therefore $J(a, b, c) = 0$ if and only if $\beta = \gamma = \lambda + \mu = 0$.

Remark. All simple three-dimensional Lie algebras over an algebraically closed field are isomorphic to sl_2.

Let the derivation D in the basis (a, b, c) have the form $D = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$.

Algebra Der(A). Let us consider the algebra

$$A = A(\lambda, \alpha, \beta, \gamma): ba = \lambda b + c,\ ca = \lambda c,\ bc = \alpha a + \beta b + \gamma c,\ \alpha\lambda \neq 0.$$

We shall now write down the right multiplication operators in this algebra:

$$R_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},\ R_b = \begin{pmatrix} 0 & -\lambda & -1 \\ 0 & 0 & 0 \\ -\alpha & -\beta & -\gamma \end{pmatrix},\ R_c = \begin{pmatrix} 0 & 0 & -\lambda \\ \alpha & \beta & \gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

We observe that the linear mapping D is a derivation if and only if

$$[R_v, D] = R_{vD}$$

for any basis element v. Writing down this equation for $v = a, b, c$, we obtain the following system of equations:

$$\begin{cases} -\alpha x_3 + \lambda y_1 + z_1 = 0,\ -\lambda x_1 - \beta x_3 + z_2 = 0,\ -x_1 - \gamma x_3 - y_2 + z_3 = 0, \\ \alpha x_2 + \lambda z_1 = 0,\ \beta x_2 = 0,\ -\lambda x_1 + \gamma x_2 - z_2 = 0, \\ \alpha x_3 - \lambda y_1 - z_1 = 0,\ \lambda x_1 + \beta x_3 - z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ -\alpha x_1 - \beta y_1 + \alpha y_2 - \gamma z_1 + \alpha z_3 = 0,\ -\alpha x_2 + \lambda z_1 - \gamma z_2 + \beta z_3 = 0, \\ -\alpha x_3 - \lambda y_1 + \gamma y_2 - \beta y_3 + z_1 = 0, \\ -\alpha x_2 - \lambda z_1 = 0,\ -\beta x_2 = 0,\ \lambda x_1 - \gamma x_2 + z_2 = 0, \\ \alpha x_1 + \beta y_1 - \alpha y_2 + \gamma z_1 - \alpha z_3 = 0,\ \alpha x_2 - \lambda z_1 + \gamma z_2 - \beta z_3 = 0, \\ \alpha x_3 + \lambda y_1 - \gamma y_2 + \beta y_3 - z_1 = 0. \end{cases}$$

By removing the obvious consequences of the equations, we obtain an equivalent system of nine equations:

$$\begin{cases} \alpha x_2 + \lambda z_1 = 0,\ \beta x_2 = 0,\ -\lambda x_1 + \gamma x_2 - z_2 = 0, \\ -\alpha x_3 + \lambda y_1 + z_1 = 0,\ -\lambda x_1 - \beta x_3 + z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ 2\lambda z_1 - \gamma z_2 + \beta z_3 = 0,\ 2\lambda y_1 - \gamma y_2 + \beta y_3 = 0, \\ -\alpha\gamma x_3 + \beta y_1 - 2\alpha y_2 - \gamma z_1 = 0. \end{cases}$$

Let us recall that $\alpha\lambda \neq 0$.

I. If $\beta \neq 0$, then $x_2 = 0$, $z_1 = 0$ and

$$\begin{cases} \lambda x_1 + z_2 = 0,\ \alpha x_3 - \lambda y_1 = 0,\ \lambda x_1 + \beta x_3 - z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ -\gamma z_2 + \beta z_3 = 0,\ 2\lambda y_1 - \gamma y_2 + \beta y_3 = 0,\ \gamma z_2 - \beta z_3 = 0, \\ -\alpha\gamma x_3 + \beta y_1 - 2\alpha y_2 = 0. \end{cases}$$

After the transformations, we have

$$\begin{cases} \lambda x_1 + z_2 = 0,\ \alpha x_3 - \lambda y_1 = 0,\ \beta x_3 - 2z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ -\alpha\gamma x_3 + \beta y_1 - 2\alpha y_2 = 0,\ \gamma z_2 - \beta z_3 = 0,\ 2\lambda y_1 - \gamma y_2 + \beta y_3 = 0. \end{cases}$$

The determinant of this system with respect to the variables $x_1, x_3, y_1, y_2, y_3, z_2, z_3$ is equal to $\alpha\beta^3(\beta + 2\lambda)$.

If the system has nontrivial solutions, then $\beta = -2\lambda$ and we obtain the following system of equations:

$$\begin{cases} \lambda x_1 + z_2 = 0,\ \alpha x_3 - \lambda y_1 = 0,\ -2\lambda x_3 - 2z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ -\alpha\gamma x_3 - 2\lambda y_1 - 2\alpha y_2 = 0,\ 2\lambda y_1 - \gamma y_2 - 2\lambda y_3 = 0, \\ \gamma z_2 + 2\lambda z_3 = 0. \end{cases}$$

Continuing with $z_2 = -\lambda x_1$, we have

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ 2\lambda x_1 - 2\lambda x_3 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ -\alpha\gamma x_3 - 2\lambda y_1 - 2\alpha y_2 = 0,\ 2\lambda y_1 - \gamma y_2 - 2\lambda y_3 = 0, \\ -\lambda\gamma x_1 + 2\lambda z_3 = 0. \end{cases}$$

Note that $x_3 = x_1$ and $\lambda\gamma x_1 = 2z_3$, $\gamma x_1 = 2z_3$, hence,

$$\begin{cases} \alpha x_1 - \lambda y_1 = 0,\ (2 + \gamma)x_1 + 2y_2 = 0, \\ \alpha\gamma x_1 + 2\lambda y_1 + 2\alpha y_2 = 0,\ 2\lambda y_1 - \gamma y_2 - 2\lambda y_3 = 0. \end{cases}$$

The rank of the fundamental matrix of this system with respect to variables x_1, y_1, y_2, y_3 is 3, since its determinant is 0, and

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 2 & 0 \\ 2\lambda & -\gamma & -2\lambda \end{vmatrix} = 4\lambda^2 \neq 0.$$

The last homogeneous system of linear equations depends on four variables and has a rank of 3, indicating it has one free variable. Thus, it is proved that the original system of equations has a unique free variable. This implies that $\dim \operatorname{Der}(A) = 1$.

II. If $\beta = 0$, we obtain the system

$$\begin{cases} \alpha x_2 + \lambda z_1 = 0,\ -\lambda x_1 + \gamma x_2 - z_2 = 0, \\ \alpha x_3 - \lambda y_1 - z_1 = 0,\ \lambda x_1 - z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ 2\lambda z_1 - \gamma z_2 = 0,\ -2\lambda y_1 + \gamma y_2 = 0, \\ -\alpha\gamma x_3 - 2\alpha y_2 + \gamma z_1 = 0. \end{cases}$$

There is no y_3 among the variables. After transformations, we have

$$\begin{cases} \alpha x_2 + \lambda z_1 = 0,\ \gamma x_2 - 2z_2 = 0, \\ \alpha x_3 - \lambda y_1 - z_1 = 0,\ \lambda x_1 - z_2 = 0,\ x_1 + \gamma x_3 + y_2 - z_3 = 0, \\ 2\lambda z_1 - \gamma z_2 = 0, \\ -\alpha\gamma x_3 - 2\alpha y_2 + \gamma z_1 = 0,\ 2\lambda y_1 - \gamma y_2 = 0. \end{cases} \quad (1)$$

A. If $\gamma \neq 0$, then the matrix P of the system with respect to the eight variables $x_1, x_2, x_3, y_1, y_2, z_1, z_2, z_3$ has a determinant equal to $\alpha\lambda^3(\gamma^2 + 4\alpha)^2$. If this determinant is non-zero, then system (1) has only the trivial solution, and $\dim \operatorname{Der}(A) = 0$.

If $4\alpha = -\gamma^2$, then the rank of matrix P is 6 since it has a non-zero minor of order 6:

$$\begin{vmatrix} 0 & 0 & \gamma^2 & 4\lambda & 0 & 4 \\ -4\lambda & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & -4\gamma & 0 & -4 & 0 \\ 0 & -\gamma^2 & 0 & 0 & 0 & 4\lambda \\ -4\lambda & 4\gamma & 0 & 0 & 0 & 0 \\ \gamma^2 & 0 & 0 & 0 & -\gamma^2 & -4\gamma \end{vmatrix} = -2^{10}\lambda^3\gamma^4.$$

Let us consider a special case with $\alpha = -1$, $\beta = 0$, $\gamma = 2$, $\lambda = 1$ (the general case is examined following the same procedure). Thus, A has the multiplication:

$$ba = b + c,\ ca = c,\ bc = -a + 2c.$$

System (1) takes the form

$$\begin{cases} -x_2 + z_1 = 0,\ x_2 - z_2 = 0, \\ x_3 + y_1 + z_1 = 0,\ x_1 - z_2 = 0,\ x_1 + 2x_3 + y_2 - z_3 = 0, \\ z_1 - z_2 = 0, \\ x_3 + y_2 + z_1 = 0,\ y_1 - y_2 = 0. \end{cases} \quad (2)$$

It is easy to verify that system (2) has a general solution:

$$x_2 = x_1,\ y_1 = y_2 = x_1 - x_3,\ z_1 = z_2 = x_1,\ z_3 = x_3,$$

where x_1, x_3 are free variables. Therefore, the general form of the derivation algebra is

$$D = \begin{pmatrix} x_1 & x_1 & x_3 \\ (-x_1 - x_3) & (-x_1 - x_3) & y_3 \\ x_1 & x_1 & x_3 \end{pmatrix}.$$

Then the algebra $\mathrm{Der}(A)$ has an additive basis of elements

$$X = e_{11} + e_{12} - e_{21} - e_{22} + e_{31} + e_{32},\ Y = e_{13} - e_{21} - e_{22} + e_{33},\ Z = e_{23}.$$

It can be readily verified that $[XY] = -2X + Y - Z$, $[XZ] = Y - Z$, $[YZ] = -2Z$. Since these specified commutators are linearly independent, $\mathrm{Der}(A)$ is isomorphic to the algebra sl_2.

B. If $\gamma = 0$, then system (1) takes the form

$$\begin{cases} \alpha x_2 + \lambda z_1 = 0,\ z_2 = 0, \\ \alpha x_3 - z_1 = 0,\ \lambda x_1 - z_2 = 0, \\ x_1 + y_2 - z_3 = 0, \\ z_1 = 0,\ y_2 = 0,\ y_1 = 0. \end{cases} \quad (3)$$

Thus, $y_1 = y_2 = 0$ и $z_1 = z_2 = 0$. Then $x_1 = x_2 = x_3 = z_3 = 0$, meaning system (3) has only the trivial solution then, $D = 0$ and $\dim \mathrm{Der}(A) = 0$.

Algebra Der$(\boldsymbol{B})$. Let us recall that

$$B = B(\lambda, \mu, \alpha, \beta, \gamma):\ ba = \lambda b,\ ca = \mu c,\ bc = \alpha a + \beta b + \gamma c,\ \alpha\lambda\mu \neq 0.$$

Let us write down the right multiplication operators in the algebra B:

$$R_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},\ R_b = \begin{pmatrix} 0 & -\lambda & 0 \\ 0 & 0 & 0 \\ -\alpha & -\beta & -\gamma \end{pmatrix},\ R_c = \begin{pmatrix} 0 & 0 & -\mu \\ \alpha & \beta & \gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

The linear mapping D is a derivation if and only if $[R_v, D] = R_{vD}$ for any basis element v. Writing down this equation for $v = a, b, c$, we obtain the following system of equations:

$$\begin{cases} -\alpha x_3 + \lambda y_1 = 0,\ -\lambda x_1 - \beta x_3 = 0,\ \lambda y_3 - \gamma x_3 - \mu y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ \beta x_2 - \lambda z_2 + \mu z_2 = 0,\ -\mu x_1 + \gamma x_2 = 0, \\ \alpha x_3 - \lambda y_1 = 0,\ \lambda x_1 + \beta x_3 = 0,\ -\lambda y_3 + \gamma x_3 + \mu y_3 = 0, \\ -\alpha x_1 - \beta y_1 + \alpha y_2 - \gamma z_1 + \alpha z_3 = 0,\ -\alpha x_2 + \lambda z_1 - \gamma z_2 + \beta z_3 = 0, \\ -\alpha x_3 - \mu y_1 + \gamma y_2 - \beta y_3 = 0, \\ -\alpha x_2 - \mu z_1 = 0,\ -\beta x_2 + \lambda z_2 - \mu z_2 = 0,\ \mu x_1 - \gamma x_2 = 0, \\ \alpha x_1 + \beta y_1 - \alpha y_2 + \gamma z_1 - \alpha z_3 = 0,\ \alpha x_2 - \lambda z_1 + \gamma z_2 - \beta z_3 = 0, \\ \alpha x_3 + \mu y_1 - \gamma y_2 + \beta y_3 = 0. \end{cases}$$

This system is equivalent to the following system of nine equations:

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ \lambda x_1 + \beta x_3 = 0,\ \gamma x_3 - \lambda y_3 + \mu y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ \beta x_2 - \lambda z_2 + \mu z_2 = 0,\ \mu x_1 - \gamma x_2 = 0, \\ \alpha x_1 + \beta y_1 - \alpha y_2 + \gamma z_1 - \alpha z_3 = 0, \\ \alpha x_2 - \lambda z_1 + \gamma z_2 - \beta z_3 = 0, \\ \alpha x_3 + \mu y_1 - \gamma y_2 + \beta y_3 = 0. \end{cases} \tag{4}$$

I. Let us assume that $\beta = 0$. Then system (4) takes the form

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ \lambda x_1 = 0,\ \gamma x_3 - \lambda y_3 + \mu y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ -\lambda z_2 + \mu z_2 = 0,\ \mu x_1 - \gamma x_2 = 0, \\ \alpha x_1 - \alpha y_2 + \gamma z_1 - \alpha z_3 = 0, \\ \alpha x_2 - \lambda z_1 + \gamma z_2 = 0, \\ \alpha x_3 + \mu y_1 - \gamma y_2 = 0. \end{cases} \tag{5}$$

1. If $\gamma = 0$, taking into account $\alpha\lambda\mu \neq 0$, we obtain

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ x_1 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ (\lambda - \mu) z_2 = 0, \\ y_2 + z_3 = 0, \\ \alpha x_2 - \lambda z_1 = 0,\ \alpha x_3 + \mu y_1 = 0. \end{cases} \tag{6}$$

A. If $\lambda = \mu$, then system (6) takes the form

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ \alpha x_2 + \lambda z_1 = 0, \\ y_2 + z_3 = 0, \\ \alpha x_2 - \lambda z_1 = 0,\ \alpha x_3 + \lambda y_1 = 0. \end{cases} \tag{7}$$

In (7) we have $x_1 = x_2 = x_3 = y_1 = z_1 = 0$, $z_3 = -y_2$. There remain three free variables y_2, y_3, z_2, indicating that the mapping has the form

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_2 & y_3 \\ 0 & z_2 & -y_2 \end{pmatrix}.$$

This implies that the algebra $\mathrm{Der}(B)$ is isomorphic to the algebra sl_2.

B. If $\lambda \neq \mu$, then system (6) takes the form

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ z_2 = 0,\ y_2 + z_3 = 0, \\ \alpha x_2 - \lambda z_1 = 0,\ \alpha x_3 + \mu y_1 = 0. \end{cases} \tag{8}$$

In (8) $y_3 = z_2 = 0$, $z_3 = -y_2$, indicating that

$$\begin{cases} y_3 = z_2 = 0,\ z_3 = -y_2, \\ \alpha x_3 - \lambda y_1 = 0,\ \alpha x_2 + \mu z_1 = 0, \\ \alpha x_2 - \lambda z_1 = 0,\ \alpha x_3 + \mu y_1 = 0. \end{cases} \tag{9}$$

It is worth noting that if B is a Lie algebra, then it is isomorphic to the algebra sl_2, and every derivation of B is inner, meaning it coincides with the right multiplication operator R_a by a suitable element $a \in B$. In this case $\mathrm{Der}(B) \cong sl_2$.

Let the algebra B not be a Lie algebra. Then, according to lemma and the conditions $\beta = \gamma = 0$ we have $\lambda + \mu \neq 0$. Then from (9) $\alpha x_3 - \lambda y_1 = 0$, $\alpha x_3 + \mu y_1 = 0$. Hence it follows $y_1 = 0$. Similarly, $z_1 = 0$. Consequently, we have

$$x_1 = x_2 = x_3 = 0,\ y_1 = y_3 = 0,\ z_1 = z_2 = 0,\ z_3 = -y_2.$$

In this case, there is one free variable y_2. This implies that the derivation algebra is one-dimensional.

2. Let $\gamma \neq 0$, then taking into account that $\alpha\gamma\lambda\mu \neq 0$, from (5) we obtain

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ x_1 = 0,\ \gamma x_3 - \lambda y_3 + \mu y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ (\lambda - \mu) z_2 = 0,\ \mu x_1 - \gamma x_2 = 0, \\ -\alpha y_2 + \gamma z_1 - \alpha z_3 = 0, \\ \alpha x_2 - \lambda z_1 + \gamma z_2 = 0, \\ \alpha x_3 + \mu y_1 - \gamma y_2 = 0. \end{cases} \tag{10}$$

A. If $\lambda = \mu$, then system (10) takes the form

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ x_1 = 0,\ \gamma x_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ \gamma x_2 = 0, \\ -\alpha y_2 + \gamma z_1 - \alpha z_3 = 0, \\ \alpha x_2 - \lambda z_1 + \gamma z_2 = 0, \\ \alpha x_3 + \mu y_1 - \gamma y_2 = 0. \end{cases}$$

It is easy to understand that this system has only the trivial solution.

B. If $\lambda \neq \mu$, then from the equations $\alpha x_2 + \mu z_1 = 0$ and $\alpha x_2 - \lambda z_1 = 0$ follows $x_2 = z_1 = 0$. Then

$$\begin{cases} x_1 = x_2 = 0,\ z_1 = z_2 = 0,\ y_2 + z_3 = 0, \\ \alpha x_3 - \lambda y_1 = 0,\ \gamma x_3 + (\mu - \lambda) y_3 = 0,\ \alpha x_3 + \mu y_1 - \gamma y_2 = 0. \end{cases}$$

It is easy to understand that all variables are expressed in terms of x_3, indicating that the algebra $\mathrm{Der}(B)$ is one-dimensional.

II. Let $\beta \neq 0$.

1. If $\gamma = 0$, then system (4) takes the form

$$\begin{cases} \alpha x_3 - \lambda y_1 = 0,\ \lambda x_1 + \beta x_3 = 0,\ (\lambda - \mu) y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ \beta x_2 - (\lambda - \mu) z_2 = 0,\ x_1 = 0, \\ \alpha x_1 + \beta y_1 - \alpha y_2 - \alpha z_3 = 0, \\ \alpha x_2 - \lambda z_1 - \beta z_3 = 0, \\ \alpha x_3 + \mu y_1 + \beta y_3 = 0. \end{cases} \tag{11}$$

Assuming that $x_1 = 0$ in the equation $\lambda x_1 + \beta x_3 = 0$, we obtain $x_3 = 0$. From the last equation in system (11), we can find $y_1 = 0$. Assuming in (11) that $x_1 = x_3 = y_1 = 0$, we have $z_3 = -y_2$. In this case, system (11) takes the form

$$\begin{cases} (\lambda - \mu) y_3 = 0, \\ \alpha x_2 + \mu z_1 = 0,\ \beta x_2 - (\lambda - \mu) z_2 = 0, \\ z_3 = -y_2, \\ \alpha x_2 - \lambda z_1 - \beta z_3 = 0, \\ \alpha x_3 + \mu y_1 + \beta y_3 = 0. \end{cases} \tag{12}$$

A. If $\lambda = \mu$ then it is easy to understand that system (12) implies the following equalities:

$$x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = z_1 = z_3 = 0,$$

meaning the solution space of system (12) is one-dimensional.

B. If $\lambda \neq \mu$, then $x_1 = x_3 = y_1 = y_3 = 0$. Thus, system (12) takes the form

$$\begin{cases} \alpha x_2 + \mu z_1 = 0, \ \beta x_2 - (\lambda - \mu) z_2 = 0, \\ y_2 + z_3 = 0, \ \alpha x_2 - \lambda z_1 - \beta z_3 = 0, \\ x_1 = x_3 = y_1 = y_3 = 0. \end{cases}$$

It can be easily verified that the rank of this system is 8, as

$$\begin{vmatrix} \alpha & 0 & \mu & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \alpha & 0 & -\lambda & -\beta \end{vmatrix} = -\beta^2\mu \neq 0.$$

It means that $D = 0$ and $\dim \operatorname{Der}(A) = 0$.

2. If $\gamma \neq 0$, then $\alpha\beta\gamma\lambda\mu \neq 0$ and it can be verified that the rank of system (4) is 8. It also means that $D = 0$ and $\dim \operatorname{Der}(A) = 0$.

Conclusions

It is proved that derivation algebras of simple three-dimensional anticommutative algebras over algebraically closed fields have limited dimension variability, and in the case when the dimension is 3, they are isomorphic to a simple three-dimensional Lie algebra. In connection with the obtained result, the question arises: what numbers can be realised in the form of the dimension of the derivation algebras of a simple anticommutative n-dimensional algebra? In particular, can the dimension of the derivation algebra be greater than the dimension of a simple anticommutative algebra?

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