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ОЦЕНИВАНИЕ ПАРАМЕТРОВ ПУАССОНОВСКОГО ПРОЦЕССА ВТОРОГО ПОРЯДКА, УПРАВЛЯЕМОГО МАРКОВСКИМ ПРОЦЕССОМ, С ИСПОЛЬЗОВАНИЕМ ПРОЦЕДУРЫ ОБНАРУЖЕНИЯ РАЗЛАДКИ

***С. Э. ВОРОБЕЙЧИКОВ*[1], *Ю. Б. БУРКАТОВСКАЯ*[1), 2)]**

[1)]*Томский государственный университет, пр. Ленина, 36, 634050, г. Томск, Россия*
[2)]*Томский политехнический университет, пр. Ленина, 30, 634050, г. Томск, Россия*

Аннотация. Рассматривается проблема оценки параметров пуассоновского процесса второго порядка, управляемого марковским процессом. Разрабатывается новый двухэтапный алгоритм их оценки. На первом этапе строятся начальные оценки параметров интенсивности с использованием метода моментов и метода максимального правдоподобия. На втором этапе эти оценки применяются в алгоритме кумулятивных сумм обнаружения разладки для определения временных интервалов, при которых интенсивность процесса остается постоянной; затем строятся оценки всех параметров. Разработанный алгоритм сравнивается с известным алгоритмом для нахождения оценок максимального правдоподобия (EM-алгоритмом). Предложенный метод не является итерационным, что обеспечивает довольно низкую вычислительную сложность.

Ключевые слова: пуассоновский процесс, управляемый марковским процессом; кумулятивные суммы; метод моментов; метод максимального правдоподобия.

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Авторы:
Сергей Эрикович Воробейчиков – доктор физико-математических наук, доцент; профессор кафедры системного анализа и математического моделирования Института прикладной математики и компьютерных наук.
Юлия Борисовна Буркатовская – кандидат физико-математических наук, доцент; доцент кафедры системного анализа и математического моделирования Института прикладной математики и компьютерных наук[1)], доцент отделения информационных технологий Инженерной школы информационных технологий и робототехники[2)].

Authors:
Sergey E. Vorobeychikov, doctor of science (physics and mathematics), docent; professor at the department of system analysis and mathematical modelling, Institute of Applied Mathematics and Computer Science.
sergeivorobeichikov@yandex.ru
https://orcid.org/0000-0002-3928-427X
Yulia B. Burkatovskaya, PhD (physics and mathematics), docent; associate professor at the department of system analysis and mathematical modelling, Institute of Applied Mathematics and Computer Science[a], and associate professor at the division for information technology, School of Computer Science and Robotics[b].
tracey@tpu.ru
https://orcid.org/0000-0002-0720-7560

ESTIMATION OF THE PARAMETERS OF A SECOND ORDER MARKOV MODULATED POISSON PROCESS USING CHANGE-POINT DETECTION PROCEDURE

S. E. VOROBEYCHIKOV[a]***, Yu. B. BURKATOVSKAYA***[a, b]

[a]*National Research Tomsk State University, 36 Lenina Avenue, Tomsk 634050, Russia*
[b]*National Research Tomsk Polytechnic University, 30 Lenina Avenue, Tomsk 634050, Russia*

Corresponding author: Yu. B. Burkatovskaya (tracey@tpu.ru)

Abstract. The problem of estimating the parameters of a second order Markov modulated Poisson process is considered. A new two-staged algorithm for their estimation is developed. In the first stage initial estimates of the intensity parameters are constructed using the method of moments and maximum likelihood method. In the second stage these estimates are used in the cumulative sums change-point detection algorithm to determine time intervals, during which the process intensity is constant; after that estimates of all parameters are constructed. The developed algorithm is compared with a well-known algorithm for finding maximum likelihood estimates (the EM algorithm). The proposed method is non-iterative, ensuring relatively low computational complexity.

Keywords: Markov modulated Poisson process; MMPP; cumulative sums; method of moments; maximum likelihood method.

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Introduction

Stochastic processes with changing parameters, in particular, processes with change points, are widely used for approximating real non-linear processes in different applications (finance, medicine, psychology, geology, literature and so on) and even in our daily lives. In the context of statistics a change point is a place or time point such that the observations follow one distribution up to that point and follow another distribution after that point. Multiple change points problems can also be defined similarly [1]. Processes with a single change point are usually used to simulate a withdraw from stable conditions or an equipment error, whereas processes with multiple change points describe real processes with several states, where switches between states occur in unknown instants. The mentioned above book is devoted to parametric change-point detection with applications to genetics, finance and medicine and contains a number of classical and recent results in this field for some commonly used distributions.

In queue theory processes with multiple change points are applied to describe queue systems with different regimes of customers arrivals. A process with two states corresponding to «usual» and «peak» time can be considered as the simplest model. These two states are characterised of different rates of arrivals: at the first case events occur in general much rarely then in the second case. When the switching times are controlled by a Markov chain and the flow of events have Poisson distribution, the model is named as Markov modulated Poisson process (MMPP). The time intervals between events can be considered as random variables obeyed to a mixture of two or more distributions.

Various modifications of the expectation maximisation (EM) algorithm based on maximum likelihood estimates are usually used to estimate mixtures of distributions. Each iteration of the algorithm consists of two steps. In the expectation step the expected value of the likelihood function is calculated using the current parameter estimate. In the maximisation step the parameters that maximise the expected maximum likelihood function are calculated. For the first time an algorithm with this name was proposed in paper [2], although iterative maximum likelihood estimates had been proposed earlier: for example, in article [3] they were used for the analysis of Markov chains. Later a large number of publications appeared in which the proposed approach was applied and modified for different classes of random processes, including processes characterised by a mixture of exponential distributions.

Markovian arrival processes have been introduced in 1970s and since then they and, in particular, MMPP have extensively been used as models for input streams to queueing systems. In paper [4] MMPP is applied to web traffic modelling. Article [5] uses MMPP to describe human inter-contact times. In work [6] a modified MMPP is applied for modelling disease dynamics based on medical claims data.

Paper [7] can be mentioned as one of the first devoted to EM estimates for MMPP. It presents a detailed description of the algorithm and some numerical examples. The author stresses that the algorithm demands some complex calculations and it should be taken into account to determine the effectiveness; nevertheless, it demands much less iterations than others. Article [8] contains a specification of the EM algorithm in order to fit an empirical counting process observed at discrete times to a Markovian arrival process. An exact numerical procedure to compute the EM steps is introduced and some examples are given. Work [9] examines maximum likelihood estimation of the second order Markovian arrival processes parameterisation of the two-state Markovian arrival processes. The problems of the starting solution and of the carrying out the numerical evaluation of the likelihood function are stated and discussed. For the starting solution the method of moments can be applied.

A review of various modifications of the EM algorithm is given in paper [10]. The positive aspects of these algorithms are due to the fact that they are based on the maximum likelihood method, which guarantees the consistency and asymptotic efficiency of estimates. However, a number of works note the disadvantages of EM algorithms. Namely, the convergence of the EM algorithm is proved under the mandatory condition of the boundedness of the log-likelihood function. Also, the presence of a large number of local maxima of the log-likelihood function leads to a large instability with respect to the initial approximation and the initial data [11]. In addition, the iterative nature of the algorithms leads to a large number of calculations, the volume of which increases with a significant number of parameters [12]. In this regard alternative approaches to estimating the parameters of mixture distributions are also being developed. In work [12] a mixture of exponential distributions is used to approximate distributions with heavy tails, which are characteristic of file lengths, call retention times, the lengths of scenes in moving picture experts group video streams and the intervals between connection requests in Internet traffic. The authors demonstrate the feasibility of approximating a distribution with heavy tails by a hyperexponential distribution and propose a recursive fitting algorithm. Firstly, the minimum intensity, which provides the highest values of the random variable and thus contributes the most to the tail of the distribution, is estimated. Secondly, this distribution is subtracted from the mixture distribution, and the intensity is estimated. Article [13] develops this approach for approximating the Pareto distribution, when estimating the average duration of the cumulative sums procedure for which an explicit formula is obtained as a result. In paper [14] the authors use a hyperexponential distribution to describe sensor networks. For estimating the parameters two original algorithms that minimise the objective function are proposed. The first algorithm consists of two stages. In the first stage the intensity parameters are estimated, in the second stage the probabilities of occurrence of observations with given intensities. Probabilities are expressed through intensities, which allows to reduce the number of estimated parameters by half at each stage. The second algorithm estimates all parameters simultaneously.

In paper [15] we consider the problem of estimating the parameters of a hyperexponential distribution with two values of the intensity. Explicit formulas for moment estimates were obtained. Also, the combined parameter estimates that use the first two sample moments in the likelihood function are proposed. In this paper we develop this approach. The properties of the combined estimates are analysed and used to evaluate MMPP parameters. In work [16] we developed a cumulative sum algorithm for Poisson process change-point detection, when parameters of the process are known. The estimates of the parameters instead of their values are used.

Problem statement

MMPP is a doubly stochastic Poisson process, the rate of which varies according to a Markov chain. We consider a process with two rates determined by the states of a non-observable controlling Markov chain. The sojourn time in state $\{j\}$, $j \in \{1, 2\}$, is determined by exponentially distributed random variables $\{x_i\}$, $i = 1, 2, \ldots$, with parameters μ_j, where x_i for odd i and for even i are distributed with different parameters.

Let us denote $x_1 + \ldots + x_m$ as T_m and $T_0 = 0$. At interval $\left[T_{i-1}, T_i\right)$ we can observe a Poisson process with the rate λ_j, where j is the state of the controlling Markov chain at the interval. The instants T_i can be considered as the change points of the observed Poisson process, where the rate of the process changes. We denote the arrival times of the observed process as t_k, $k = 1, 2, \ldots, t_0 = 0$.

The problem is to estimate parameters $\{\lambda_1, \lambda_2, \mu_1, \mu_2\}$ from observations t_k.

Parameter estimation

We consider the process $\{\tau_i\}_{i \geq 1}$, where $\tau_i = t_i - t_{i-1}$ is the length of the i interval between arriving events in the observed flow. The values $\{\tau_i\}$ for MMPP are independent and exponentially distributed with one of two possible parameters. It allows us to use the hyperexponential distribution as a model for the observed data. The hyperexponential distribution has the density function

$$f(\tau) = p\lambda_1 e^{-\lambda_1 \tau} + (1-p)\lambda_2 e^{-\lambda_2 \tau},\ p \in (0, 1), \tag{1}$$

which is a mixture of two exponential distributions.

To estimate parameters λ_1, λ_2 and p of the hyperexponential distribution we can apply either the method of moments (using three sample moments) or the maximum likelihood method. In the latter case maximisation of the likelihood relation demands the use of numerical methods with the prescribed accuracy.

We propose combined estimates on the basis of both mentioned above methods. Let us have a sample $\{\tau_1, \ldots, \tau_N\}$ obeyed distribution (1). Primarily, we calculate the following two first moments:

$$m_1 = \frac{1}{N}\sum_{k=1}^{N}\tau_k,\ m_2 = \frac{1}{N}\sum_{k=1}^{N}\tau_k^2.$$

Then, equating them to the corresponding theoretical moments, we have the system of equations

$$\begin{cases} m_1 = \dfrac{p}{\lambda_1} + \dfrac{1-p}{\lambda_2}, \\ m_2 = \dfrac{2p}{\lambda_1^2} + \dfrac{2(1-p)}{\lambda_2^2}. \end{cases}$$

Let us introduce the following notations:

$$x = \frac{p}{\lambda_1},\ y = \frac{1-p}{\lambda_2}. \tag{2}$$

We obtain the system of equations

$$\begin{cases} m_1 = x + y, \\ m_2 = \dfrac{2x^2}{p} + \dfrac{2y^2}{1-p}. \end{cases}$$

Expressing from the first equation $y = m_1 - x$ and substituting it in the second equation, we come to the quadratic equation

$$x^2 - 2m_1px + p\left(m_1^2 - \frac{1-p}{2}m_2\right) = 0.$$

If the inequality $m_2 - 2m_1^2 \geq 0$ holds true, the equation has two solutions:

$$x_{1,2} = pm_1 \pm \sqrt{p(1-p)}\sqrt{\frac{m_2}{2} - m_1^2}.$$

Choosing the largest root, we have

$$x = pm_1 + \sqrt{p(1-p)}\sqrt{\frac{m_2}{2} - m_1^2},$$

$$y = (1-p)m_1 - \sqrt{p(1-p)}\sqrt{\frac{m_2}{2} - m_1^2}.$$

Then using the notations (2), values λ_1, λ_2 are defined as functions of the parameter p:

$$\begin{aligned} \hat{\lambda}_1 &= \frac{1}{m_1 + \sqrt{\dfrac{1-p}{p}}\sqrt{\dfrac{m_2}{2} - m_1^2}}, \\ \hat{\lambda}_2 &= \frac{1}{m_1 - \sqrt{\dfrac{p}{1-p}}\sqrt{\dfrac{m_2}{2} - m_1^2}}. \end{aligned} \tag{3}$$

It should be noted that $\hat{\lambda}_1 < \hat{\lambda}_2$. If we take the smallest root for x, it obtains the same estimators for λ_1, λ_2, except of $\hat{\lambda}_2 < \hat{\lambda}_1$.

As both rates should be positive, parameter p should obey the following inequality:

$$m_1 - \sqrt{\frac{p}{1-p}}\sqrt{\frac{m_2}{2} - m_1^2} > 0.$$

It means that

$$p < \frac{2m_1^2}{m_2}.$$

Then we define the log-likelihood function

$$L(p) = \sum_{n=1}^{N} \log\left(p\lambda_1 e^{-\lambda_1\tau_n} + (1-p)\lambda_2 e^{-\lambda_2\tau_n} \right),$$

where parameters $\{\lambda_1, \lambda_2\}$ are calculated according to equations (3). Maximising the function $L(p)$ on p, $0 < p < \frac{2m_1^2}{m_2}$, and using relations of the equations (3), we obtain the estimates of parameters $\{\lambda_1, \lambda_2\}$.

In paper [16] we developed a cumulative sum algorithm for Poisson process change point detection, when parameters $\{\lambda_1, \lambda_2\}$ are known. To construct algorithm we use statistics and cumulative sums from work [16]. Statistics z_i is defined as the logarithm of the likelihood ratio for exponential distribution

$$z_i = \ln\left(\frac{\lambda_2 \exp(-\lambda_2\tau_i)}{\lambda_1 \exp(-\lambda_1\tau_i)} \right) = \ln\frac{\lambda_2}{\lambda_1} - (\lambda_2 - \lambda_1)\tau_i. \tag{4}$$

Let $\lambda(t)$ be the rate of the arrival process at moment t. We introduce the following two hypothesis:

$$H_j = \{\lambda(t) = \lambda_j,\ j = 1, 2\}.$$

We suppose that $\lambda_1 < \lambda_2$ and denote $\delta = \frac{\lambda_2}{\lambda_1}$, where $\delta > 1$. Statistics z_i in distribution (4) have the form

$$z_i = \ln\delta - \lambda_1(\delta - 1)\tau_i.$$

Under hypothesis H_j, these statistics have the following properties:

$$E\left[z_i | H_1 \right] = \ln\delta - (\delta - 1) < 0,$$
$$E\left[z_i | H_2 \right] = \ln\delta - \left(\frac{\delta - 1}{\delta} \right) > 0.$$

Note that the mean of statistics z_i changes the sign with the change the rate of the process.

In our algorithm we use estimators (3) obtained above instead of the parameter values. We introduce positive values $h^{(1)}$, $h^{(2)}$ as the algorithms thresholds and construct cumulative sum S_i, which is recalculated at the instants of arrivals:

$$\begin{gathered} S_0 = 0,\ S_i = \begin{cases} \max\{0,\ S_{i-1} + c_{i-1}z_i\}, \text{ if } S_{i-1} + c_{i-1}z_i < h_i,\ i \ge 1, \\ 0, \text{ if } S_{i-1} + c_{i-1}z_i \ge h_i,\ i \ge 1, \end{cases} \\ z_i = \ln\frac{\hat\lambda_2}{\hat\lambda_1} - \left(\hat\lambda_2 - \hat\lambda_1 \right)\tau_i, \\ c_0 = 1,\ c_i = \begin{cases} c_{i-1}, \text{ if } S_{i-1} + c_{i-1}z_i < h_i,\ i \ge 1, \\ -c_{i-1}, \text{ if } S_{i-1} + c_{i-1}z_i \ge h_i,\ i \ge 1, \end{cases} \\ h_i = \begin{cases} h^{(1)}, \text{ if } c_{i-1} = 1, \\ h^{(2)}, \text{ if } c_{i-1} = 1. \end{cases} \end{gathered} \tag{5}$$

If $c_i = 1,$ the sum detects the increase of the rate, i. e. the change of the hypothesis from H_1 to H_2, when it reaches threshold $h^{(1)}$. If $c_i = -1$, the sum detects the decrease of the rate, i. e. the change of the hypothesis from H_2 to H_1, when it reaches threshold $h^{(2)}$.

Let sequence $\{\sigma_m\}_{m \geq 0}$ be the sequence of instants, when the cumulative sum reaches the threshold h_i, i. e.

$$\sigma_0 = 0,\ \sigma_m = \min\left\{t_i > \sigma_{m-1} : S_i \geq h\right\},\ m \geq 1. \tag{6}$$

Consider sequence $\{n_m\}_{m \geq 0}$ associated with sequence $\{\sigma_m\}_{m \geq 0}$ as follows:

$$n_0 = 0,\ n_m = \max\left\{t_i \leq \sigma_m : S_i > 0,\ S_{i-1} = 0\right\},\ m \geq 1. \tag{7}$$

Thus, instant n_m is the first instant, when the cumulative sum becomes positive to reach then the threshold.

The algorithm for change-point detection is described as follows. Calculate the cumulative sum given by equation (5). Then construct sequences $\{\sigma_m\}$, $\{n_m\}$ defined by equations (6) and (7). If for the last instant n_m one has $m = 2l + 1$, one sets $n_{2l+2} = t_N$. If for the last instant n_m one has $m = 2l$, one sets $n_{2l+1} = t_N$. Here the odd instants n_{2l+1} are the estimators of the instants, when the rate changes from λ_1 to λ_2, and the even instants n_{2l+2} are the estimators of the instants when the rate changes from λ_2 to λ_1. So, we consider the Markov chain to be in the first state at intervals $\left[n_{2l},\ n_{2l+1}\right]$ and in the second state at intervals $\left[n_{2l+1},\ n_{2l+2}\right]$.

According to sequential change-point detection algorithms two types of errors are considered: the false alarm and the skipping of the change. A false alarm occurs, when one of the cumulative sums reaches the corresponding threshold in the case of the constant intensity of the arrival process. A skipping of the change occurs, when the change of the parameter occurs but the corresponding cumulative sum did not reach its threshold. Usual characteristics of any sequential algorithm are the mean delay and the mean length of the interval between false alarms, as well as the probabilities of a false alarm and of a skipping the change.

When implementing the algorithm, it is possible to encounter false alarm situations. We shall record all the exceeding the thresholds by either the first or the second cumulative sum. If the same sum reaches threshold several times in a row, we only record the first occurrence.

Now we are ready to estimate parameters $\{\mu_1,\ \mu_2\}$ of the controlling Markov chain and parameters $\{\lambda_1,\ \lambda_2\}$. Let N_1 be the number of intervals $\left[q_{2l},\ q_{2l+1}\right]$ and T_1 be the total length of these intervals. Also, let N_2 be the number of intervals $\left[q_{2k+1},\ q_{2k+2}\right]$ and T_2 be the total length of these intervals. The estimates for $\{\mu_1,\ \mu_2\}$ have the following form:

$$\hat{\mu}_j = \frac{N_j}{T_j},\ j = 1, 2. \tag{8}$$

Let M_1 be the number of events at intervals $\left[q_{2l},\ q_{2l+1}\right]$. Also, let M_2 be the number of events at intervals $\left[q_{2k+1},\ q_{2k+2}\right]$. Thus, we can estimate rates $\{\lambda_1,\ \lambda_2\}$ as reciprocal values to the mean length of the interval between neighbouring events, when the controlling chain is in the corresponding state:

$$\hat{\lambda}_j = \frac{M_j}{T_j},\ j = 1, 2. \tag{9}$$

Simulation results and their discussion

To analyse the algorithm simulations were conducted. The results are represented in table 1. Here $\{\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2\}$ are the process parameters, $\left\{\hat{\lambda}_1,\ \hat{\lambda}_2,\ \hat{\mu}_1,\ \hat{\mu}_2\right\}$ are the estimates (8) and (9), h_1, h_2 are the thresholds of the change-point detection algorithm. The parameters in first two rows were chosen according to paper [7], where they were used to implement EM algorithm.

Table 2 represents the simulation results from article [7]. Here I is the number of iterations, other notations are the same as in table 1. Columns for estimates $\left\{\hat{\lambda}_1,\ \hat{\lambda}_2,\ \hat{\mu}_1,\ \hat{\mu}_2\right\}$ have two rows, the first one contains initial values of parameters $\{\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2\}$, which are necessary for EM algorithm. It should be mentioned that they are not calculated in the frame of the algorithm.

Table 1

Simulation results

λ_1	λ_2	$\hat{\lambda}_1$	$\hat{\lambda}_2$	μ_1	μ_2	$\hat{\mu}_1$	$\hat{\mu}_2$	h_1	h_2
50	100	46.72	105.96	2.0	5.0	2.160	4.970	1.0	4.7
10	100	9.95	106.39	1.0	10.0	1.010	10.610	2.5	4.1
50	100	49.46	102.19	1.0	2.0	1.10	2.070	1.3	4.4
1	2	0.98	2.16	0.04	0.1	0.038	0.090	1.0	5.3
1	10	1.02	10.63	0.10	1.0	0.097	1.050	2.2	4.1
1	3	0.96	3.13	0.04	0.1	0.045	0.096	2.5	2.9

Table 2

Simulation results for EM algorithm

λ_1	λ_2	$\hat{\lambda}_1$	$\hat{\lambda}_2$	μ_1	μ_2	$\hat{\mu}_1$	$\hat{\mu}_2$	I
50	100	53.9	128	2	5	1.90	10.12	26
50	100	50.8	105	2	5	2.94	5.26	26

The results demonstrate a good quality of the proposed approach. The error of the estimates is similar to EM estimates but the procedure is much simpler because it is not iterative.

Conclusions

A new algorithm for estimation of the parameters of a second order Markov modulated Poisson process is proposed. Simulation results demonstrate the algorithm quality, which is similar to widely used EM algorithm. The proposed approach does not demand the initial estimates for all parameters and it is not iterative. Hence, the computational complexity of the algorithm is much less than the computational complexity of the EM method. We are going to develop our approach for arbitrary order MMPP.

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