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О ХОРОШО *ve*-УКРЫТЫХ И ХОРОШО *ev*-УКРЫТЫХ ГРАФАХ

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Аннотация. Исследуются классы хорошо *ve*-укрытых графов и хорошо *ev*-укрытых графов. Граф называется хорошо *ve*-укрытым (соответственно хорошо *ev*-укрытым), если все его независимые минимальные *ve*-доминирующие (соответственно независимые минимальные *ev*-доминирующие) множества имеют одинаковую мощность. Показано, что задачи распознавания рассматриваемых классов графов являются co-NP-полными даже для некоторых сужений этих классов. Найдены характеризации в терминах запрещенных порожденных подграфов для максимальных наследственных подклассов хорошо *ve*-укрытых графов и хорошо *ev*-укрытых графов. Установлена вычислительная сложность задач, связанных с наименьшими независимыми *ve*-доминирующими и наименьшими независимыми *ev*-доминирующими множествами.

Ключевые слова: граф; независимое *ve*-доминирующее множество; независимое *ev*-доминирующее множество; хорошо *ve*-укрытый граф; хорошо *ev*-укрытый граф; наследственный класс графов; NP-полнота.

ON WELL *ve*-COVERED AND WELL *ev*-COVERED GRAPHS

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Abstract. In this paper, the classes of well *ve*-covered graphs and well *ev*-covered graphs are investigated. A graph is called well *ve*-covered (well *ev*-covered, respectively) if all its minimal independent *ve*-dominating (minimal independent *ev*-dominating, respectively) sets have the same cardinality. It is shown that the problems of recognising the graph classes under consideration are co-NP-complete even for some restrictions of these classes. Characterisations, in terms of forbidden induced subgraphs, for the maximal hereditary subclasses of well *ve*-covered graphs and well *ev*-covered graphs are found. The computational complexity of problems related to minimum independent *ve*-dominating and minimum independent *ev*-dominating sets is established.

Keywords: graph; independent *ve*-dominating set; independent *ev*-dominating set; well *ve*-covered graph; well *ev*-covered graph; hereditary graph class; NP-completeness.

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Introduction

In this paper, we use graph-theoretic terminology of J. Bondy and U. Murty [1] (unless noted otherwise), and computational complexity terminology of M. Garey and D. Johnson [2]. For concepts related to domination in graphs, we follow T. Haynes et al. [3].

Throughout this paper G is a simple (i. e. a finite, undirected, loopless and without multiple edges) graph with the vertex set $V(G)$ and the edge set $E(G)$. A class of graphs is a set of graphs closed under isomorphism. A graph is trivial if it has only one vertex. A graph with no edges is called empty. An edge of G is two-element subset of $V(G)$. Often in the paper, an edge $\{u, v\}$ will be written uv or vu, dropping the curly brackets. The vertices u and v are called end-vertices of the edge uv.

Let G be a graph. For any vertex $v \in V(G)$, the open neighbourhood of v, denoted by $N(v)$, is defined by $\{u \in V(G) \mid uv \in E(G)\}$. The closed neighbourhood of v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. The neighbourhood of an edge $e = uv \in E(G)$ is defined as $N(e) = N(u) \cup N(v)$. Clearly, $\{u, v\} \subseteq N(e)$. The degree of v is defined as $|N(v)|$. A vertex of degree 0 is called isolated. We say that a graph is isolate-free if it has no isolated vertices.

As usual, by P_n we denote the chordless path on n vertices, by C_n the chordless cycle on n vertices, and by $K_{1,n}$ the star on $n+1$ vertices, i. e. the complete bipartite graph with partition classes of cardinalities 1 and n. Given two graphs – G_1 and G_2, we denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2. In particular, for a positive integer n, the disjoint union of n copies of a graph G is denoted by nG. The complement of a graph G is denoted by $\overline{G}$.

For a graph G and a non-empty subset S of $V(G)$, the subgraph $G(S)$ of G induced by S has S as its vertex set and two vertices – u and v – are adjacent in $G(S)$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an induced subgraph if there is a non-empty subset S of $V(G)$ such that $H = G(S)$. For a (finite or infinite) set M of graphs, a graph G is called M-free if no induced subgraph of G is isomorphic to a graph in M.

As usual, the term «minimal» («maximal», respectively) with respect to the sets having some property means an inclusion-wise minimal (an inclusion-wise maximal, respectively) set with this property, while the terms «minimum» and «maximum» refer to the cardinality of a set with a given property.

A subset $S \subseteq V(G)$ is an independent set of G if no two vertices in S are adjacent. A subset $D \subseteq V(G)$ is a dominating set of G if each vertex of $V(G) \setminus D$ is adjacent to a vertex of D. If, in addition, D is an independent set, then D is an independent dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. The independent domination number $i(G)$ of G is the minimum cardinality of an independent dominating set in G, or equivalently, is the minimum cardinality of a maximal independent set in G. The independence number $\alpha(G)$ of G is the maximum cardinality of an independent set in G.

It is well known that for any graph G, $\gamma(G) \le i(G) \le \alpha(G)$. A graph is well-covered if all its maximal independent sets have the same cardinality, equivalently if every maximal independent set is maximum. In other words, a graph G is well-covered, if and only if $i(G) = \alpha(G)$ is satisfied. The concept of a well-covered graph was introduced by M. Plummer [4] in 1970. These graphs are of interest because, whereas the independent set problem is NP-complete for general graphs, the maximum independent set can be found efficiently in a well-covered graph by using a simple greedy algorithm.

The problem of recognising well-covered graphs is known to be co-NP-complete [5; 6]. It remains co-NP-complete even for $K_{1,4}$-free graphs [7], for circulant graphs [8], and for $\{2P_5, K_{1,10}\}$-free squares of line graphs [9]. However, the problem is solvable in polynomial time for $K_{1,3}$-free graphs [10; 11], for graphs with girth at least 5 [12], for graphs that contain neither C_4 nor C_5 as a subgraph [13], for graphs with a bounded maximum degree [14], or for chordal graphs [15].

The following two variants of domination in graphs, which are called vertex-edge domination and edge-vertex domination, were introduced by K. Peters in work [16] and further studied in papers [17–19]. A vertex $u \in V(G)$ is said to ve-dominate an edge $xy \in E(G)$ if either (1) $u = x$ or $u = y$, that is, u is incident to xy or (2) ux or uy is an edge in G, that is, u is incident to an edge that is adjacent to xy. In other words, a vertex u ve-dominates the edges incident to vertices in $N[u]$. An edge $uw \in E(G)$ is said to ev-dominate a vertex $x \in V(G)$ if either (1) $u = x$ or $w = x$, that is, uw is incident to x or (2) ux or wx is an edge in G, that is, x is adjacent to u or w. In other words, an edge uw ev-dominates the vertices in $N(uw)$.

We need some additional concepts about private edges and private vertices (see [20–22]). Let G be a graph and let S be a subset of the vertex set $V(G)$ of G. A vertex $u \in S$ has a private edge $xy \in E(G)$ (with respect to S), if (1) u *ve*-dominates the edge xy (i. e. u is incident to xy or u is adjacent to either x or y), and (2) no other vertex in S *ve*-dominates xy (i. e. for all vertices $w \in S \setminus \{u\}$, w is not incident to xy and w is not adjacent to either x or y). It was shown in papers [20; 21] that a *ve*-dominating set S of a non-trivial connected graph G is minimal if and only if each vertex $u \in S$ has at least one private edge (with respect to S).

Now let F be a subset of the edge set $E(G)$ of G. An edge $uw \in F$ has a private vertex $x \in V(G)$ (with respect to F), if (1) uw *ev*-dominates the vertex x (i. e. uw is incident to x or x is adjacent to either u or w), and (2) no other edge in F *ev*-dominates w (i. e. for all edges $ab \in F \setminus \{uw\}$, ab is neither incident to x and neither a nor b is adjacent to x). It was shown in paper [22] that an *ev*-dominating set F of a non-trivial connected graph G is minimal if and only if each edge $uw \in F$ has at least one private vertex (with respect to F).

The vertex-edge domination number (or simply the *ve*-domination number) $\gamma_{ve}(G)$ is the minimum cardinality of a *ve*-dominating set of G. An independent vertex-edge dominating set (or simply independent *ve*-dominating set) is a vertex subset that is both independent and *ve*-dominating. The independent vertex-edge domination number (or simple independent *ve*-domination number) $i_{ve}(G)$ is the minimum cardinality of an independent *ve*-dominating set, and the upper independent vertex-edge domination number (or simply upper independent *ve*-domination number) $\alpha_{ve}(G)$ is the maximum cardinality taken over all minimal independent *ve*-dominating sets of G.

The edge-vertex domination number (or simply the *ev*-domination number) $\gamma_{ev}(G)$ is the minimum cardinality of an *ev*-dominating set of G. An independent edge-vertex dominating set (or simply independent *ev*-dominating set) is an edge subset that is both independent (i. e. a matching) and *ev*-dominating. The independent edge-vertex domination number (or simple independent *ev*-domination number) $i_{ev}(G)$ is the minimum cardinality of an independent *ev*-dominating set, and the upper independent edge-vertex domination number (or simply upper independent *ev*-domination number) $\alpha_{ev}(G)$ is the maximum cardinality taken over all minimal independent *ev*-dominating sets of G.

For any graph G, the above vertex-edge domination parameters are related by the following chain of inequalities established in paper [17]: $\gamma_{ve}(G) \le i_{ve}(G) \le \alpha_{ve}(G)$. Let us remark that the empty graph O_n on $n \ge 1$ vertices has a unique minimal *ve*-dominating set, namely, the empty set. So, $\gamma_{ve}(O_n) = i_{ve}(O_n) = \alpha_{ve}(O_n) = 0$.

Note that only graphs without isolated vertices have *ev*-dominating sets. It is well-known (see [17]) that if G is an isolate-free graph, then $\gamma_{ev}(G) \le i_{ev}(G) \le \alpha_{ev}(G)$. Moreover, as shown in paper [17], the equality $\gamma_{ev}(G) = i_{ev}(G)$ between edge-vertex domination parameters $\gamma_{ev}(G)$ and $i_{ev}(G)$ holds for all isolate-free graphs G.

A graph G is well *ve*-covered if $i_{ve}(G) = \alpha_{ve}(G)$, i. e. all its minimal independent *ve*-dominating sets have the same cardinality $\alpha_{ve}(G)$. This concept was introduced by R. Boutrig et al. [21]. They proved that the problem of recognising well *ve*-covered graphs is co-NP-complete and presented a constructive characterisation of well *ve*-covered trees.

In order to characterise the trees T with $i_{ev}(T) = \alpha_{ev}(T)$, R. Boutrig and M. Chellali [22] introduced the concept of the well *ev*-covered graph, which is similar to the concept of well *ve*-covered graph but in terms of the independent *ev*-domination. A graph G is well *ev*-covered if $i_{ev}(G) = \alpha_{ev}(G)$, i. e. all its minimal independent *ev*-dominating sets have the same cardinality $\alpha_{ev}(G)$. In paper [22], R. Boutrig and M. Chellali proposed some open problems concerning well *ev*-covered graphs, among them:

1. Can you recognise a well *ev*-covered graph in polynomial time?
2. Characterise the well *ev*-covered graphs.

It has been shown by M. Chellali, in his recent paper [23], that the problem of recognising well *ev*-covered graphs is co-NP-complete. This result shows that the property of being not well *ev*-covered is NP-complete. Thus, returning to the open problems formulated in article [22], one can note that the existence of a good characterisation of the well *ev*-covered graphs is very unlikely.

The remaining part of the present paper is organised as follows.

In the section «Well *ve*-covered graphs», we prove that the problem of recognising well *ve*-covered graphs is co-NP-complete even when restricted to the class of $K_{1,4}$-free graphs. This result strengthens the result given in paper [21], where it is shown that the problem is co-NP-complete for general graphs. We also prove that, under the same restriction, the problem of recognising a graph that has minimal independent *ve*-dominating sets of at most t distinct cardinalities is co-NP-complete for any given integer $t \ge 2$.

The section «Well *ev*-covered graphs» contains the proof that the problem of recognising well *ev*-covered graphs is still co-NP-complete even for some graph classes defined by certain forbidden induced subgraphs. Note that our polynomial transformation in the proof of this result is simpler than that in paper [23] and allow us to reduce the problem from general graphs to connected graphs in a proper subclass of $\{C_4, \overline{K_1 \cup C_4}, K_{1,5}, 2P_4\}$-free graphs.

In the section «Hereditary properties», we show that the classes of well *ve*-covered graphs and well *ev*-covered graphs are not hereditary, and therefore, they have no characterisations in terms of forbidden induced subgraphs. As an alternative, we provide finite characterisations for the maximal hereditary subclasses of these classes in terms of forbidden induced subgraphs.

In the section «Related results», we apply the previous results to obtain new cases where the decision problems associated with the computation of $i_{ve}(G)$, $i_{ev}(G)$ and $\gamma_{ev}(G)$ are NP-complete. Also in this section we introduce a concept of a well paired-dominated graph and study the computational complexity of recognising such graphs.

The section «Conclusions» provides a summary of our results and presents some open problems.

In what follows all decision problems are stated in the standard instance – question format [2].

Well *ve*-covered graphs

In this section we shall consider the following decision problem.

Non-Well *ve*-Covered Graph problem

Instance: a graph G.

Question: is it true that the graph G is not well *ve*-covered? In other words, are there two minimal independent *ve*-dominating sets of distinct cardinalities in G?

We first prove that the Non-Well *ve*-Covered Graph problem is an NP-complete problem and thereby establish that recognising the class of well *ve*-covered graphs is a co-NP-complete problem. Then we extend this result by showing that recognising a graph that has minimal independent *ve*-dominating sets of at most t distinct cardinalities is a co-NP-complete problem for any $t \geq 2$. In our proof, we use a transformation from the following well-known NP-complete decision problem (see [2]).

Three-Dimensional Matching (3-DM) problem

Instance: pairwise disjoint sets X, Y, Z of some finite cardinality k, and a collection M of three-element sets, where each member of M includes exactly one element from each of X, Y and Z.

Question: is there a set of pairwise disjoint members of M, whose union is $X \cup Y \cup Z$?

We say that the instance of 3-DM problem is solvable if there exists a set of pairwise disjoint members of M, whose union is $X \cup Y \cup Z$.

Theorem 1. *The Non-Well ve-Covered Graph problem is NP-complete for $K_{1,4}$-free graphs.*

P r o o f. It can be checked in polynomial time whether a *ve*-dominating set is minimal, since the property of being a *ve*-dominating set is superhereditary (for more details see [3]). Thus, it is obvious that the Non-Well *ve*-Covered Graph problem is a member of NP since we can guess subsets $D_1, D_2 \subseteq V(G)$ and, in polynomial time, verify that D_1 and D_2 are minimal independent *ve*-dominating sets of distinct cardinalities.

To establish NP-completeness we start by describing how to construct an instance of the Non-Well *ve*-Covered Graph problem from one for the 3-DM problem. Let $I = (X, Y, Z, M)$ be an instance of the 3-DM problem, where $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Let $\{a, b_i, c_i \mid i = 1, 2, \ldots, k\}$ be a set of $2k+1$ new elements, each of which does not belong to the union $X \cup Y \cup Z$. We define $M_x = \{\{x_i, a\} \mid i = 1, 2, \ldots, k\}$ and $M_y = \{\{y_i, b_i\}, \{y_i, c_i\} \mid i = 1, 2, \ldots, k\}$. Let $G(I)$ be the intersection graph of the subset system $S = M \cup M_x \cup M_y \cup \{\{a\}\}$, i. e. the vertex set of $G(I)$ is S, and two vertices are adjacent if and only if the corresponding subsets have a non-empty intersection. It is easy to see that the construction of the graph $G(I)$ can be accomplished in polynomial time.

An example of graph $G(I)$ constructed for $k = 3$ and

$$M = \{\{x_1, y_2, z_1\}, \{x_3, y_2, z_3\}, \{x_2, y_1, z_1\}, \{x_1, y_2, z_3\}, \{x_3, y_1, z_2\}, \{x_2, y_3, z_1\}\}$$

is given in fig. 1, where (p, q, r) stands for the vertex $\{x_p, y_q, z_r\}$.

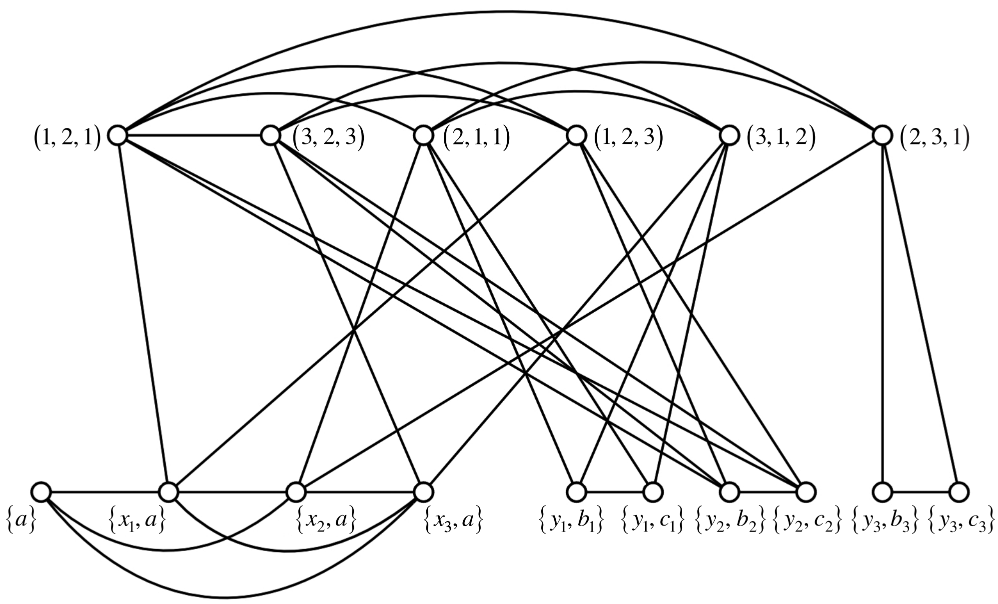

Fig. 1. An example of the graph $G(I)$

Note that the graph $G(I)$ is connected and has always a minimal independent *ve*-dominating set of cardinality $k+1$, for example $\{\{a\}\} \cup \{\{y_i, b_i\} \mid i = 1, 2, \ldots, k\}$ is such a set.

To prove the theorem, it is sufficient to show that the following three claims are valid.

Claim 1. The graph $G(I)$ is $K_{1,4}$-free.

P r o o f. Indeed, since $|T| \leq 3$ for each $T \in S$, any four sets, each intersecting with *T*, cannot be pairwise disjoint. This, in turn, means that the graph $G(I)$ cannot contain a star $K_{1,4}$ centred at vertex *T* as an induced subgraph. The claim is proved.

Claim 2. For any minimal independent *ve*-dominating set *D* of $G(I)$, the inequalities $k \leq |D| \leq k+1$ hold.

P r o o f. Let *D* be an arbitrary minimal independent *ve*-dominating set of $G(I)$. Since any $k+2$ members of the subset system *S* obviously contain an intersecting pair, we have that any independent (in particular, minimal) *ve*-dominating set of $G(I)$ contains at most $k+1$ vertices, and hence $|D| \leq k+1$.

For $i = 1, 2, \ldots, k$, define

$$S_i = \{\{y_i, b_i\}, \{y_i, c_i\}, \{x_p, y_i, z_r\} \mid \{x_p, y_i, z_r\} \in M,\ 1 \leq p, r \leq k\}.$$

Let e_i be the edge of the graph $G(I)$ connecting vertices $\{y_i, b_i\}$ and $\{y_i, c_i\}$, for $1 \leq i \leq k$. Clearly, the edge e_i can only be *ve*-dominated by vertices from S_i. Since the set S_i is a clique of the graph $G(I)$, we have that $|D \cap S_i| = 1$, for each $i = 1, 2, \ldots, k$. Thus, $|D| \geq k$, and the claim is proved.

Claim 3. The instance $I = (X, Y, Z, M)$ is solvable for the 3-DM problem if and only if the graph $G(I)$ has a minimal independent *ve*-dominating set of cardinality *k*.

P r o o f. Let the instance $I = (X, Y, Z, M)$ is solvable for the 3-DM problem, i. e. *M* contains a subset M' of *k* pairwise disjoint members, whose union is equal to $X \cup Y \cup Z$. Construct a set $D \subseteq V(G(I))$ of cardinality *k* by including in it all the vertices of the graph $G(I)$ that correspond to elements of M'. It is easy to see that *D* is independent and *ve*-dominating set of $G(I)$. The minimality of the set *D* follows from the fact that every vertex $\{x_p, y_q, z_r\} \in D$ has the edge $\{\{y_q, b_q\}, \{y_q, c_q\}\} \in E(G(I))$ as a private edge (with respect to *D*).

Assume now that the graph $G(I)$ has a minimal independent *ve*-dominating set *D* of cardinality *k*. For $1 \leq i \leq k$, let e_i be the edge of $G(I)$ between the vertices $\{y_i, b_i\}$ and $\{y_i, c_i\}$, and let S_i be the set defined as in the proof of claim 2. Then as in the proof of claim 2, we have for each $i = 1, 2, \ldots, k$ that $|D \cap S_i| = 1$, and therefore, the intersection of *D* with the set $M_x \cup \{\{a\}\}$ is empty. Suppose that $\{y_i, b_i\} \in D$ for some $i \in \{1, 2, \ldots, k\}$. Then $|D \cap M| < k$, and hence there exists an element $x_j \in X$, which does not belong to any member of $D \cap M$. But then no vertex in *D*, which *ve*-dominates the edge $\{\{x_j, a\}, \{a\}\} \in E(G(I))$, a contradiction since *D* is a *ve*-dominating set of $G(I)$. Thus, the intersection of D with the set M_y is also empty, and consequently $D \subseteq M$. Since D is an independent *ve*-dominating set its vertices correspond to exactly k pairwise disjoint members of M, whose union is $X \cup Y \cup Z$. Therefore, the instance I is solvable. This completes the proof of the claim.

So, we have obtained a polynomial transformation from the 3-DM problem to the Non-Well *ve*-Covered Graph problem within $K_{1,4}$-free graphs, and the theorem is proved.

Let $W_{ve}(t)$ be the class of graphs, each of which has minimal independent *ve*-dominating sets of at most t cardinalities. Note that $W_{ve}(1)$ is the class of well *ve*-covered graphs and $W_{ve}(t) \subset W_{ve}(t+1)$ holds for $t \geq 1$. Consider the following decision problem.

Non-*t*-Well *ve*-Covered Graph problem
Instance: a graph G and an integer number $t \geq 2$.
Question: is it true that $G \notin W_{ve}(t)$? In other words, are there $t+1$ minimal independent *ve*-dominating sets of distinct cardinalities in G?

Based on theorem 1, the following result is obtained.

Theorem 2. *For any $t \geq 2$, the Non-t-Well ve-Covered Graph problem is NP-complete even for $K_{1,4}$-free graphs.*

Proof. It is not hard to see that the Non-*t*-Well *ve*-Covered Graph problem belongs to NP. To prove the NP-completeness, we will use a polynomial transformation from the Non-Well *ve*-Covered Graph problem, which is NP-complete by theorem 1. Let G be a non-trivial connected $K_{1,4}$-free graph, which constitutes an instance for the Non-Well *ve*-Covered Graph problem. Consider the graph H, which is the disjoint union of t components $G_1, G_2, \ldots, G_t$, each of which is isomorphic to the graph G. It is easy to see that the graph H is $K_{1,4}$-free, and that H can be constructed in polynomial time in the number of vertices of G. To complete the proof of the theorem, it suffices to show the following.

Claim 4. $G \in W_{ve}(1)$ if and only if $H \in W_{ve}(t)$.

Proof. Assume that G is a well *ve*-covered graph, i. e. $G \in W_{ve}(1)$. Let D_1 and D_2 be any two minimal independent *ve*-dominating sets of the graph H. The intersection of D_i, $i \in \{1, 2\}$, with the vertex set $V(G_j)$ of the j^{th} component G_j in H is a minimal independent *ve*-dominating set of the graph G_j. Since G_j is isomorphic to the graph G and all minimal independent *ve*-dominating sets in G have the same cardinality, we conclude that $|D_1| = |D_2|$. Hence, $H \in W_{ve}(1)$, and since $W_{ve}(1) \subset W_{ve}(t)$ for $t \geq 2$, we obtain $H \in W_{ve}(t)$.

For the converse implication assume that $H \in W_{ve}(t)$ for some $t \geq 2$. Suppose on the contrary that the graph G is not well *ve*-covered. Then there are minimal independent *ve*-dominating sets D_1 and D_2 in G with $|D_1| \neq |D_2|$. Without loss of generality let $|D_1| < |D_2|$. For $i = 0, 1, \ldots, t$, define the set S_i by

$$S_i = \left(D_1^1 \cup D_1^2 \cup \ldots \cup D_1^i\right) \cup \left(D_2^{i+1} \cup D_2^{i+2} \cup \ldots \cup D_2^t\right),$$

where D_1^j and D_2^j are copies of the sets D_1 and D_2 in the j^{th} component G_j of the graph H, respectively. Clearly, for each $i \in \{0, 1, \ldots, t\}$, the set S_i is a minimal independent *ve*-dominating set of H and $|S_i| = i|D_1| + (t-i)|D_2|$. Therefore, for $0 \leq k < l \leq t$, we obtain $|S_k| - |S_l| = (|D_2| - |D_1|)(l-k)$ and, consequently, $|S_0| > |S_1| > \ldots > |S_t|$. But this contradicts the fact that $H \in W_{ve}(t)$, and the claim is proved.

So, we have obtained a polynomial transformation from the Non-Well *ve*-Covered Graph problem restricted to $K_{1,4}$-free graphs to the Non-*t*-Well *ve*-Covered Graph problem restricted to $K_{1,4}$-free graphs, and the theorem is proved.

Well *ev*-covered graphs

In this section we shall consider the following decision problem.

Non-Well *ev*-Covered Graph problem
Instance: a graph G.
Question: is it true that the graph G is not well *ev*-covered? In other words, are there two minimal independent *ev*-dominating sets of distinct cardinalities in G?

We prove that the Non-Well *ev*-Covered Graph problem is an NP-complete problem and thereby establish that recognising the class of well *ev*-covered graphs is a co-NP-complete problem. In our proof, we use a polynomial transformation from the following well-known NP-complete decision problem (see [2]).

Three-Satisfiability (3Sat) problem

Instance: a collection $C = \{c_1, c_2, \ldots, c_m\}$ of m clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of n Boolean variables such that $|c_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: is there a truth assignment for X that simultaneously satisfies all the clauses in C?

Recall that a truth assignment for X is a mapping $\varphi : X \to \{0, 1\}$ that assigns a value $\varphi(x_i) \in \{0, 1\}$ to each variable $x_i \in X$. We extend φ to the set $X \cup \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ of the literals over X by setting $\varphi(\overline{x}_i) = \overline{\varphi(x_i)}$, $i = 1, 2, \ldots, n$. A literal $\ell \in X \cup \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ is true under φ if $\varphi(\ell) = 1$. A clause over X is a disjunction of some literals. A truth assignment φ for X satisfies a clause $c_j \in C$ if c_j involves at least one true literal under φ. A collection C of clauses over X is satisfiable if and only if there exists some truth assignment for X that simultaneously satisfies all the clauses in C.

In order to formulate the following result, we need to introduce special notations for some particular classes of graphs: A is the class of $\{\overline{K_1 \cup C_4}, K_{1,5}, 2P_4\}$-free graphs, and B is the class of $\{C_4, C_6, C_7, C_8, C_9\}$-free graphs. In fact, it is not hard to see that any graph in $A \cap B$ is also C_p-free and $\overline{C}_q$-free for each $p \geq 10$ and $q \geq 6$, as well as P_s-free and $\overline{P}_t$-free for each $s \geq 9$ and $t \geq 5$.

Theorem 3. *The Non-Well ev-Covered Graph problem is NP-complete in the class of graphs $A \cap B$.*

Proof. It can be checked in polynomial time whether an *ev*-dominating set is minimal, since the property of being an *ev*-dominating set is superhereditary (for more details see [3]). Thus, the Non-Well *ev*-Covered Graph problem is in NP since we can guess subsets $E_1, E_2 \subseteq E(G)$ and, in polynomial time, verify that E_1 and E_2 are minimal independent *ev*-dominating sets of distinct cardinalities. To show that it is NP-complete, we construct a polynomial transformation from the 3Sat problem. Without loss of generality, we may assume that each variable appears (as x or $\overline{x}$) in at least one clause and no clause in C contains a variable x_i and its negation $\overline{x}_i$ because such a clause is satisfied by any truth assignment and can be eliminated. We also assume that all clauses are distinct and $m = |C| \geq 2$. The proof uses ideas from paper [9].

Given an instance $J = (C, X)$ of the 3Sat problem, construct the following graph $G(J)$. Some vertices of this graph are associated with the parameters of the 3Sat problem and use the same notation, and some vertices are auxiliary. Specifically, the vertex set of the graph $G(J)$ is the union $C' \cup X'$, where $C' = \{c_j \mid j = 1, 2, \ldots, m\}$ and $X' = \{x_i, y_i, z_i, \overline{x}_i \mid i = 1, 2, \ldots, n\}$. Edges of $G(J)$ are defined as follows.

1. The set C' induces a clique.

2. The set X' induces n vertex-disjoint stars $K^i_{1,3}$, $i = 1, 2, \ldots, n$, each one with four vertices – x_i, y_i, z_i and $\overline{x}_i$ – and the edges $x_i y_i$, $y_i z_i$ and $y_i \overline{x}_i$.

3. For each clause $c_j = \left(\ell^1_j \vee \ell^2_j \vee \ell^3_j\right)$, $j = 1, 2, \ldots, m$, introduce the three edges – $c_j \ell^1_j$, $c_j \ell^2_j$ and $c_j \ell^3_j$ – between C' and X' in $G(J)$.

The graph $G(J)$ can obviously be constructed in time polynomial in m and n. Figure 2 shows an example of graph $G(J)$ when $X = \{x_1, x_2, x_3, x_4\}$ and $C = \{c_1, c_2, c_3, c_4, c_5\}$, where $c_1 = (x_1 \vee x_2 \vee x_3)$, $c_2 = (\overline{x}_1 \vee x_2 \vee x_3)$, $c_3 = (x_1 \vee x_3 \vee \overline{x}_4)$, $c_4 = (\overline{x}_2 \vee \overline{x}_3 \vee x_4)$, $c_5 = (\overline{x}_1 \vee \overline{x}_3 \vee x_4)$.

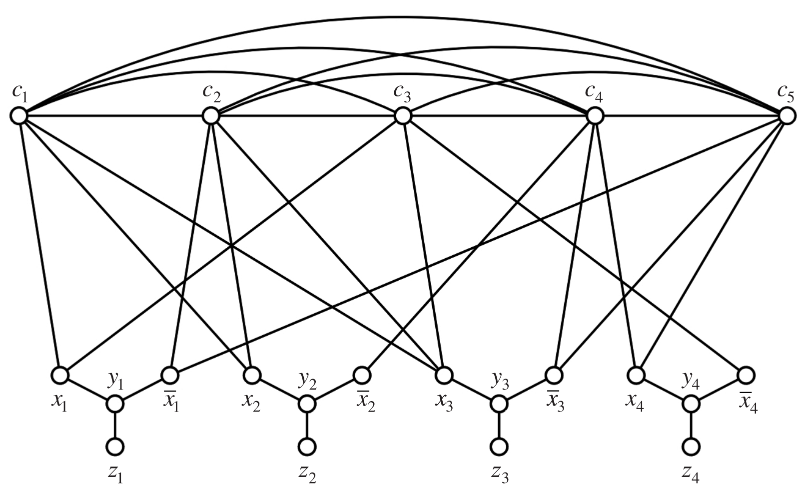

Fig. 2. An example of the graph $G(J)$

Note that the graph $G(J)$ is connected and has always a minimal independent ev-dominating set of cardinality $n+1$, for example $\{c_1c_2\} \cup \{y_iz_i \mid i=1, 2, \ldots, n\}$ is such a set. Recall that $m \geq 2$.

To prove the theorem, it is sufficient to show that the following three claims are valid.

Claim 5. The graph $G(J)$ belongs to the class $A \cap B$.

P r o o f. It is not difficult to show that the graph $G(J)$ does not contain any graph in $\{\overline{K_1 \cup C_4}, K_{1,5}, 2P_4\} \cup \cup \{C_4, C_6, C_7, C_8, C_9\}$ as an induced subgraph. Hence, $G(J)$ is in the class $A \cap B$, as required. The claim is proved.

Let us denote $X_i = \{x_iy_i, y_iz_i, y_i\overline{x}_i\}$, $i=1, 2, \ldots, n$.

Claim 6. For any minimal independent ev-dominating set E' in $G(J)$, the following statements hold:

(i) $|E' \cap X_i| = 1$ for $i=1, 2, \ldots, n$;

(ii) $|E' \cap EN(C')| \leq 1$, where $EN(C')$ is the set of all edges in $G(J)$ that have at least one end-vertex in C';

(iii) $|E'| \in \{n, n+1\}$.

P r o o f. (i) If $E' \cap X_i = \varnothing$ for some $i \in \{1, 2, \ldots, n\}$, then the vertex z_i cannot be ev-dominated. Therefore, $|E' \cap X_i| \geq 1$ for each $i=1, 2, \ldots, n$. On the other hand, since E' is an independent set of edges and, for each $i=1, 2, \ldots, n$, the set X_i forms the star $K_{1,3}^i$ in $G(J)$, we have $|E' \cap X_i| \leq 1$, $i=1, 2, \ldots, n$. Thus, $|E' \cap X_i| = 1$ for $i=1, 2, \ldots, n$.

(ii) Suppose on the contrary that the intersection $E' \cap EN(C')$ contains two edges – e_1 and e_2. Let u be a private vertex (with respect to E') of the edge e_1. Suppose that $u \in X'$, say $u \in V(K_{1,3}^i)$ for some $i \in \{1, 2, \ldots, n\}$. Then by the statement (i) there exists the edge $e' \in E' \cap X_i$ such that e' is distinct from e_1, and e' ev-dominates u, which is impossible. Hence, $u \in C'$. Finally, since C' is a clique in $G(J)$ and $e_2 \in E' \cap EN(C')$, we conclude that e_2 ev-dominates u, again a contradiction. This contradiction shows that $|E' \cap EN(C')| \leq 1$, as required.

(iii) By the statement (i), the set E' contains exactly n edges of $G(X')$. Thus, $|E'| \geq n$. The fact that $|E'| \leq n+1$ follows from the statement (ii).

The claim is proved.

Claim 7. C is satisfiable if and only if the graph $G(J)$ has a minimal independent ev-dominating set of cardinality n.

P r o o f. First, suppose that there exists a truth assignment φ satisfying C. We construct a set $E' \subset \bigcup_{i=1}^{n} X_i$ by choosing edges that correspond to the true literals. Namely, for each $i=1, 2, \ldots, n$, do the following. If $\varphi(x_i)=1$, then the edge x_iy_i is included in E'. On the other hand, if $\varphi(x_i)=0$, then the edge $y_i\overline{x}_i$ is included in E'. Thus, E' is of cardinality n.

We shall prove that E' is a minimal independent ev-dominating set in $G(J)$. Indeed, since $X_i \cap X_j = \varnothing$ (for $1 \leq i \neq j \leq n$), we have that E' is a matching, i. e. an independent set of edges. Since φ satisfies C, each vertex of C' is adjacent to an end-vertex of an edge in E', and hence is ev-dominated by such an edge. Since for each $i=1, 2, \ldots, n$ the set X_i form a star $K_{1,3}$ in $G(J)$, we have that each vertex of X' is either incident to an edge in E' or is adjacent to an end-vertex of an edge in E', and hence is ev-dominated by such an edge. Therefore, E' is an ev-dominating set in $G(J)$. Finally, the minimality of the set E' follows directly from the fact that for each $i=1, 2, \ldots, n$ the vertex z_i is a private vertex of the unique edge in E' that belongs to the set X_i.

Conversely, assume that E' is a minimal independent ev-dominating set of cardinality n in the graph $G(J)$. By claim 6 (i), we have that $|E' \cap X_i| = 1$ for $i=1, 2, \ldots, n$. Therefore, $E' \subset E(G(X'))$ and E' covers at most one of the vertices $x_i, \overline{x}_i$ for each $i=1, 2, \ldots, n$. Since E' ev-dominates every vertex in C', we can define a partial truth assignment φ' satisfying C by deciding a literal $\ell_i \in \{x_i, \overline{x}_i\}$ to be true if the corresponding vertex of C' is ev-dominated by the edge $\ell_iy_i \in E' \cap X_i$. It remains to extend, if necessary, φ' to a full assignment. This completes the proof of the claim.

So, we have obtained a polynomial transformation from the 3Sat problem to the Non-Well ev-Covered Graph problem restricted to the class of graphs $A \cap B$, and the theorem is proved.

Let $W_{ev}(t)$ be the class of graphs, each of which has minimal independent ev-dominating sets of at most t cardinalities. Note that $W_{ev}(1)$ is the class of well ev-covered graphs and $W_{ev}(t) \subset W_{ev}(t+1)$ holds for $t \geq 1$. Consider the following decision problem.

Non-*t*-Well *ev*-Covered Graph problem

Instance: a graph G and an integer number $t \geq 2$.

Question: is it true that $G \notin W_{ev}(t)$? In other words, are there $t+1$ minimal independent ev-dominating sets of distinct cardinalities in G?

The following result can be obtained by applying theorem 3 and the same approach that we have used to prove theorem 2.

Theorem 4. *For any $t \geq 2$, the Non-t-Well ev-Covered Graph problem is NP-complete even in the class of graphs $A \cap B$.*

Hereditary properties

Let us remind basic notions related to hereditary properties of graphs referring for more details to the monograph by S. Kitaev and V. Lozin (see [24, chap. 2]).

A class of graphs is called hereditary if every induced subgraph of a graph in this class also belongs to the class. The family of hereditary classes includes many important classes of graphs, for instance, bipartite graphs, chordal graphs, planar graphs, line graphs, graphs of vertex degree at most k (for a fixed k), etc.

It is well known that a class X of graphs is hereditary if and only if X is M-free (i. e. every $G \in X$ is M-free) for some set M. In this case graphs in M are called forbidden induced subgraphs for the class X. Generally, for a hereditary class X the set M of forbidden induced subgraphs is not unique. For instance, M can be defined as the set of all graphs that are not in X. However, to describe X it is sufficient to find only so-called minimal forbidden induced subgraphs. A forbidden induced subgraph H for hereditary class X is minimal if H is not in X but every proper induced subgraph of H is in X. It is known that for every hereditary class the set of minimal forbidden induced subgraphs exists and is unique.

As noted in paper [25], every (hereditary or not) class X of graphs can be «approximated» by two hereditary classes as follows: by $\lfloor X \rfloor$ we denote the maximal hereditary subclass contained in X and by $\lceil X \rceil$ the minimal hereditary class containing X. It is known that $\lfloor X \rfloor$ and $\lceil X \rceil$ are uniquely defined and that $\lfloor X \rfloor \subseteq X \subseteq \lceil X \rceil$ with equalities holding if and only if X is a hereditary class.

In order to prove that the classes $W_{ve}(1)$ and $W_{ev}(1)$ of well ve-covered graphs and well ev-covered graphs are not hereditary classes, let us prove the following result.

Theorem 5. *For any graph H, there exists a graph G that contains H as an induced subgraph and G is both a well ve-covered graph and a well ev-covered graph.*

P r o o f. Given a graph H with the vertex set $V(H) = \{u_1, u_2, \ldots, u_n\}$, $n \geq 1$, construct a new graph G as follows. For each vertex u_i, $i = 1, 2, \ldots, n$, add two new vertices – a_i and b_i, and join u_i to a_i and a_i to b_i, respectively. More precisely,

$$V(G) = V(H) \cup \{a_i, b_i \mid i = 1, 2, \ldots, n\} \text{ and } E(G) = E(H) \cup \{u_i a_i, a_i b_i \mid i = 1, 2, \ldots, n\}.$$

Clearly, by the construction, H is an induced subgraph of G.

Let D be any minimal independent ve-dominating set of G, and let $U = D \cap V(H)$. Then for each vertex $u_i \in V(H) \setminus U$, the set D contains exactly one of the two vertices – a_i or b_i, since otherwise the edge $a_i b_i$ cannot be ve-dominated. Therefore, $|D| = |U| + |V(H) \setminus U| = n$, and hence G is well ve-covered.

Now let F be any minimal independent ev-dominating set of G. It is easy to see that for each $i = 1, 2, \ldots, n$, the set F contains exactly one of the two edges – $u_i a_i$ or $a_i b_i$, since otherwise the vertex b_i cannot be ev-dominated. By minimality of F, we have $F \cap E(H) = \varnothing$. Therefore, $|F| = n$, and hence G is well ev-covered.

The theorem is proved.

By theorem 5 we conclude that both hereditary classes – $\lceil W_{ve}(1) \rceil$ and $\lceil W_{ev}(1) \rceil$ – are coincide with the class of all graphs.

Now we consider a hereditary subclass of the class $W_{ve}(1)$. A graph G is called perfectly well ve-covered if every induced subgraph of G is well ve-covered. We characterise perfectly well ve-covered graphs in terms of minimal forbidden induced subgraphs.

In the sequel of this section, the notation $y \sim z$ ($y \nsim z$, respectively) means that the vertices y and z are adjacent (non-adjacent, respectively). For disjoint sets of vertices Y and Z, the notation $Y \sim Z$ ($Y \nsim Z$, respectively) means that every vertex of Y is adjacent (non-adjacent, respectively) to every vertex of Z. In the case when $Y = \{y\}$, we also write $y \sim Z$ and $y \nsim Z$ instead of $\{y\} \sim Z$ and $\{y\} \nsim Z$, respectively. Also, for the sake of simplicity, a subgraph of G induced by a vertex set $\{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ will be denoted by $G(u_1, u_2, \ldots, u_k)$ instead of $G(\{u_1, u_2, \ldots, u_k\})$.

Theorem 6. *For a graph G, the following statements are equivalent:*

(i) G is a perfectly well ve-covered graph;

(ii) G is a $\{G_1, G_2\}$*-free graph, where the graphs* G_1 *and* G_2 *are shown in fig. 3.*

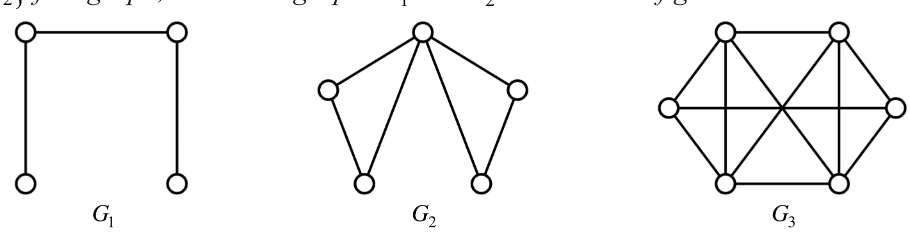

Fig. 3. The graphs G_1, G_2 and G_3

Proof. We prove first the implication (i) $\Rightarrow$ (ii). It is easy to check that $i_{ve}(H)=1$ and $\alpha_{ve}(H)=2$ for $H\in\{G_1, G_2\}$. Therefore, both graphs – G_1 and G_2 – are not perfectly well *ve*-covered graphs, and they are minimal. It follows that none of them can be an induced subgraph of a perfectly well *ve*-covered graph.

Now we prove the converse implication (ii) $\Rightarrow$ (i). Suppose that G is a $\{G_1, G_2\}$-free graph, but G is not a perfectly well *ve*-covered graph. We may assume that G is minimal, i. e. each proper induced subgraph of G is perfectly well *ve*-covered. By definition, there are two minimal independent *ve*-dominating sets – D_1 and D_2 – in G with $|D_1|\neq|D_2|$. Without loss of generality, we may assume that $|D_1|>|D_2|$. Since D_1, D_2 are minimal *ve*-dominating sets and $|D_1\setminus D_2|>|D_2\setminus D_1|$, there are three distinct vertices – $u_1, u_2\in D_1\setminus D_2$, $w\in D_2\setminus D_1$ – and there are two distinct edges – ab, cd – in $E(G)$, such that the following properties are satisfied: 1) ab and cd are private edges of u_1 and u_2 (with respect to D_1), respectively; 2) w *ve*-dominates both edges – ab and cd.

Claim 8. $u_1\notin\{c, d\}, u_2\notin\{a, b\}$ and $u_1\nsim\{c, d\}, u_2\nsim\{a, b\}$.

Proof. This is obvious from the definition of a private edge.

In all claims below we use the fact that $u_1\nsim u_2$, since D_1 is an independent set.

Claim 9. The edges ab and cd are not adjacent in G.

Proof. Assume, to the contrary, that ab and cd are adjacent. Without loss of generality, let us suppose that $b=c$. By claim 8, $u_1\nsim\{c, d\}$ and $u_2\nsim\{a, b\}$. Hence, $b\nsim\{u_1, u_2\}$. Since u_1 *ve*-dominates ab and u_2 *ve*-dominates cd, we have that $u_1\sim a$ and $u_2\sim d$. If $a\nsim d$, then $\{a, b, d, u_1\}$ induces G_1, a contradiction. Thus, $a\sim d$. But then $\{a, d, u_1, u_2\}$ induces G_1. This contradiction completes the proof of the claim.

Claim 10. The vertex w is not incident to any edge in $\{ab, cd\}$.

Proof. Assume, to the contrary, that w is incident to some edge in $\{ab, cd\}$. By claim 9, edges ab and cd are not adjacent in G. Hence, taking into account the symmetry, we may assume without loss of generality that $w=b$. By symmetry, there are four cases depending on whether or not vertices u_1 and u_2 belong to $\{a, b\}$ and $\{c, d\}$, respectively.

Case 1. $u_1=a$, $u_2=c$. Since w *ve*-dominates cd, we have $b\sim d$. Taking into account claim 8, we conclude that $a\nsim d$ and $b\nsim c$. Thus, $\{a, b, c, d\}$ induces G_1, a contradiction.

Case 2. $u_1=a$, $u_2\notin\{c, d\}$. By claim 8, $a\nsim\{c, d\}$ and $u_2\nsim\{a, b\}$. Since u_2 *ve*-dominates cd, we may assume without loss of generality that $u_2\sim c$. Then $b\nsim c$, otherwise $\{a, b, c, u_2\}$ induces G_1, which is a contradiction. Since w *ve*-dominates cd, we have that $b\sim d$. But then $\{a, b, c, d\}$ induces G_1, a contradiction.

Case 3. $u_1\notin\{a, b\}$, $u_2=c$. By claim 8, $u_1\nsim\{c, d\}$ and $c\nsim\{a, b\}$. Since u_1 *ve*-dominates ab, we may assume without loss of generality that $u_1\sim a$. Since w *ve*-dominates cd, we have $b\sim d$. Then $a\sim d$, otherwise $\{a, b, c, d\}$ induces G_1, which is a contradiction. But now $\{a, c, d, u_1\}$ induces G_1, a contradiction.

Case 4. $u_1\notin\{a, b\}$, $u_2\notin\{c, d\}$. By claim 8, $u_1\nsim\{c, d\}$ and $u_2\nsim\{a, b\}$. Since u_2 *ve*-dominates cd, u_2 must have a neighbour in the set $\{c, d\}$. Due to symmetry, we may assume without loss of generality that $u_2\sim c$. Since u_1 *ve*-dominates ab, u_1 must have a neighbour in the set $\{a, b\}$.

Suppose that $u_1\sim a$. Then $a\nsim c$, otherwise $\{a, c, u_1, u_2\}$ induces G_1, which is a contradiction. Since w *ve*-dominates cd, w must have a neighbour in the set $\{c, d\}$. If $w\sim c$, then $\{a, b, c, u_2\}$ induces G_1, a contradiction. Hence, $w\nsim c$ and, as consequence, $w\sim d$. But then G contains either induced $G_1=G(a, b, c, d)$ (if $a\nsim d$) or induced $G_1=G(a, c, d, u_1)$ (if $a\sim d$), a contradiction.

Now let us suppose that $u_1\nsim a$. Hence, $u_1\sim b$. But then G contains either induced $G_1=G(b, c, u_1, u_2)$ (if $w\sim c$) or induced $G_1=G(b, c, d, u_1)$ (if $w\nsim c$, $w\sim d$), a contradiction. Claim 10 is proved.

Claim 11. The vertices u_1 and u_2 are not incident to edges ab and cd, respectively.

Proof. By claim 9, edges ab and cd are not adjacent in G, and by claim 10, $w\notin\{a, b, c, d\}$. Assume, to the contrary, that u_1 is incident to ab or u_2 is incident to cd. Due to symmetry, there are two cases depending on whether or not the vertices u_1 and u_2 belong to $\{a, b\}$ and $\{c, d\}$, respectively.

Case 1. $u_1=a$, $u_2=c$. In this case $b\nsim d$, otherwise $\{a, b, c, d\}$ induces G_1 a contradiction. Since w *ve*-dominates both edges – ab and cd, w has at least one neighbour in each of $\{a, b\}$ and $\{c, d\}$.

Suppose that $w\nsim\{b, d\}$. Then $w\sim\{a, c\}$ and, as consequence, $G_1=G(a, b, c, w)$ and $G_1=G(a, c, d, w)$, a contradiction. Hence, w has a neighbour in the set $\{b, d\}$. Due to symmetry, we may assume, without loss of generality, that $w\sim b$. Let $w\nsim d$. Then clearly $w\sim c$ and, as consequence, $\{b, c, d, w\}$ induces G_1, a contradiction.

Thus, $w \sim \{b, d\}$. Next, we conclude that $w \sim \{a, c\}$, since otherwise $G_1 = G(a, b, d, w)$ or $G_1 = G(b, c, d, w)$, which again gives a contradiction. But now, the set $\{a, b, c, d, w\}$ induces G_2, a contradiction.

Case 2. $u_1 = a$, $u_2 \notin \{c, d\}$. Since u_2 *ve*-dominates cd, u_2 must have a neighbour in the set $\{c, d\}$. Due to symmetry, we may assume without loss of generality that $u_2 \sim c$. In this case $b \nsim c$, otherwise $\{a, b, c, u_2\}$ induces G_1, a contradiction. If $b \sim d$, then $\{a, b, c, d\}$ induces G_1, a contradiction. Hence, $b \nsim d$. By considering the subgraph $H = G(a, b, c, d, w)$, one can prove that H contains either G_1 or G_2 as an induced subgraph. The proof is similar to that of case 1 of this claim and hence omitted. Claim 11 is proved.

We now consider the general case, in which a, b, c, d, u_1, u_2, w are seven distinct vertices. Since u_1 *ve*-dominates ab, we may assume without loss of generality that $u_1 \sim a$. Due to symmetry, we may assume without loss of generality that $u_2 \sim c$ (since u_2 *ve*-dominates cd). By claim 8, $u_1 \nsim \{c, d\}$ and $u_2 \nsim \{a, b\}$. If $a \sim c$, then $\{a, c, u_1, u_2\}$ induces G_1, a contradiction. Hence, $a \nsim c$. Since w *ve*-dominates both edges – ab and cd, w has at least one neighbour in each of $\{a, b\}$ and $\{c, d\}$.

If $w \nsim \{b, d\}$, then $w \sim \{a, c\}$. In this case we conclude that $w \sim \{u_1, u_2\}$, since otherwise $G_1 = G(a, c, u_1, w)$ or $G_1 = G(a, c, u_2, w)$, which gives a contradiction. But now, the set $\{a, c, u_1, u_2, w\}$ induces G_2, a contradiction. Hence, w has a neighbour in the set $\{b, d\}$. Due to symmetry, we may assume, without loss of generality, that $w \sim b$. If $w \sim c$, then $w \nsim a$, otherwise we have the previous situation when $w \sim \{a, c\}$. But then G contains either induced $G_1 = G(a, b, c, u_2)$ (if $b \sim c$) or induced $G_1 = G(a, b, c, w)$ (if $b \nsim c$), a contradiction.

Thus, we have $w \sim \{b, d\}$ and $w \nsim \{a, c\}$. If $b \sim c$ then $\{a, b, c, u_2\}$ induces G_1, a contradiction. Hence, $b \nsim c$. Similarly we have $a \nsim d$, otherwise $\{a, c, d, u_1\}$ induces G_1, a contradiction. Finally, we conclude that G contains either induced $G_1 = G(a, b, c, d)$ (if $b \sim d$) or induced $G_1 = G(a, b, d, w)$ (if $b \nsim d$). This contradiction completes the proof of the claim.

Thus, we have contradictions in all possible cases, and the proof of the implication (ii) $\Rightarrow$ (i) is complete. Theorem 6 is proved.

By theorem 6, $W_{ve}(1)$ contains the class of $\{G_1, G_2\}$-free graphs. On the other hand, any hereditary subclass of $W_{ve}(1)$ is contained in the class of $\{G_1, G_2\}$-free graphs, since graphs G_1 and G_2 do not belong to $W_{ve}(1)$. Thus, we obtain the following corollary.

Corollary 1. *The unique maximal hereditary subclass of* $W_{ve}(1)$ *is the class of* $\{G_1, G_2\}$*-free graphs.*

Finally, we consider a hereditary subclass of the class $W_{ev}(1)$. A graph G is called perfectly well *ev*-covered if every isolate-free induced subgraph of G is well *ev*-covered. The following result can be obtained using a similar technique as in the proof of theorem 6.

Theorem 7. *For a graph G, the following statements are equivalent:*

(i) G is a perfectly well ev-covered graph;

(ii) G is a $\{G_1, G_2, G_3\}$*-free graph, where the graphs* G_1, G_2 *and* G_3 *are shown in fig. 3.*

By theorem 7, $W_{ev}(1)$ contains the class of $\{G_1, G_2, G_3\}$-free graphs. On the other hand, any hereditary subclass of $W_{ev}(1)$ is contained in the class of $\{G_1, G_2, G_3\}$-free graphs, since graphs G_1, G_2 and G_3 do not belong to $W_{ev}(1)$. Thus, we obtain the following corollary.

Corollary 2. *The unique maximal hereditary subclass of* $W_{ev}(1)$ *is the class of* $\{G_1, G_2, G_3\}$*-free graphs.*

As a simple consequence of theorems 6 and 7, we have the following corollary.

Corollary 3. *The classes of perfectly well ve-covered graphs and perfectly well ev-covered graphs are recognisable in polynomial time.*

Related results

Consider the following decision problem.

Independent *ve*-Dominating Set problem
Instance: a graph G and an integer k.
Question: is $i_{ve}(G) \leq k$?

The Independent *ve*-Dominating Set problem is known to be NP-complete for general graphs. Moreover, it remains NP-complete even for graphs having a specific structure, such as bipartite graphs and chordal graphs [17]. On the other hand, the problem can be solved in polynomial time for trees [17].

A graph G is said to be 2-*ve*-indominated if there exists an integer $k \geq 1$ such that $|D| \in \{k, k+1\}$ for every minimal independent *ve*-dominating set D in G. The proof of theorem 1 implies the following result.

Theorem 8. *The Independent ve-Dominating Set problem is an NP-complete problem for 2-ve-indominated $K_{1,4}$-free graphs.*

P r o o f. The problem of deciding «$i_{ve}(G) \leq k$» belongs to NP. Let $I = (X, Y, Z, M)$ be an instance of the 3-DM problem. Set $k = |X| = |Y| = |Z|$. For the $K_{1,4}$-free graph $G(I)$ in the proof of theorem 1, we have $i_{ve}(G(I)) \geq k$ and $\alpha_{ve}(G(I)) = k+1$. Hence, $G(I)$ is a 2-*ve*-indominated graph. Moreover, the instance $I = (X, Y, Z, M)$ is solvable for the 3-DM problem if and only if $i_{ve}(G(I)) = k$. Theorem 8 is proved.

Now consider the following decision problem.

Independent *ev*-Dominating Set problem

Instance: a graph G and an integer k.

Question: is $i_{ev}(G) \leq k$?

The Independent *ev*-Dominating Set problem is known to be NP-complete for general graphs. Moreover, it remains NP-complete for bipartite graphs [17]. On the other hand, the problem can be solved in polynomial time for trees [17].

A graph G is said to be 2-*ev*-indominated if there exists an integer $k \geq 1$ such that $|E'| \in \{k, k+1\}$ for every minimal independent *ev*-dominating set E' in G. The proof of theorem 3 implies the following result.

Theorem 9. *The Independent ev-Dominating Set problem is an NP-complete problem for 2-ev-indominated graphs in the class of graphs $A \cap B$.*

P r o o f. The problem of deciding «$i_{ev}(G) \leq k$» belongs to NP. Let $J = (C, X)$ be an instance of the 3Sat problem. Set $k = n = |X|$. For the graph $G(J) \in A \cap B$ in the proof of theorem 3, we have $i_{ev}(G(J)) \geq k$ and $\alpha_{ev}(G(J)) = k+1$. Hence, $G(J)$ is a 2-*ev*-indominated graph. Moreover, there exists a satisfying truth assignment for C if and only if $i_{ev}(G(J)) = k$. Theorem 9 is proved.

Since $\gamma_{ev}(G) = i_{ev}(G)$ for any isolate-free graph G (see [17]), we can use theorem 9 to show that the problem of deciding «$\gamma_{ev}(G) \leq k$» is NP-complete in the class of graphs $A \cap B$.

Consider the concept of paired-domination in graphs, which was introduced by T. Haynes and P. Slater [26; 27] (see [28] for a survey).

A subset $D \subseteq V(G)$ is a paired-dominating set of G if every vertex of $V(G)$ is adjacent to some vertex in D and the induced subgraph $G(D)$ contains a perfect matchings (not necessarily induced). Note that every graph without isolated vertices has a paired-dominating set. It is easy to see that for any paired-dominating set D of G, the cardinality $|D|$ is even. A paired-dominating set D in G is minimal if no proper subset of D is a paired-dominating set. The paired domination number of G, denoted by $\gamma_{\mathrm{pr}}(G)$, is the minimum cardinality of a paired-dominating set of G. The upper paired domination number of G, denoted by $\Gamma_{\mathrm{pr}}(G)$ is the maximum cardinality of a minimal paired-dominating set of G.

The following characterisation of a minimal paired-dominating set is obtained in work [29].

Proposition 1. *Let G be an isolate-free graph and D be a paired-dominating set of G. Then D is a minimal paired-dominating set if and only if any two vertices – $x, y \in D$ – satisfy one of the following conditions:*

(i) the induced subgraph $G(D \setminus \{x, y\})$ does not contain a perfect matching;

(ii) without loss of generality, x is an end-vertex in $G(D)$ adjacent to y;

(iii) there exists a vertex $u \in V(G) \setminus D$ such that $N(u) \cap D \subseteq \{x, y\}$.

The following result can be found in paper [23].

Proposition 2. *If G is an isolate-free graph, then* $2i_{ev}(G) = \gamma_{\mathrm{pr}}(G) \leq \Gamma_{\mathrm{pr}}(G) \leq 2\alpha_{ev}(G)$.

We call a graph well paired-dominated if all its minimal paired-dominating sets are of the same cardinality (cf. [30] for the difference in definitions). That is, all minimal paired-dominating sets are also minimum paired-dominating sets. In other words, a graph G is well paired-dominated, if and only if $\gamma_{\mathrm{pr}}(G) = \Gamma_{\mathrm{pr}}(G)$ is satisfied.

Consider the following decision problem.

Non-Well Paired-Dominated Graph problem

Instance: a graph G.

Question: is it true that the graph G is not well paired-dominated? In other words, are there two minimal paired-dominating sets of distinct cardinalities in G?

Using theorem 3, one can show that the Non-Well Paired-Dominated Graph problem is an NP-complete problem.

Theorem 10. *The Non-Well Paired-Dominated Graph problem is NP-complete in the class of graphs* $A \cap B$.

Proof. It can be checked in polynomial time whether a paired-dominating set is minimal. This follows from the characterisation of a minimal paired-dominating set given in proposition 1 and the fact that the general perfect matching problem can be solved in polynomial time (see, for example, [31]). Thus, the Non-Well Paired-Dominated Graph problem is in NP since we can guess subsets $D_1, D_2 \subseteq V(G)$ and, in polynomial time, verify that D_1 and D_2 are minimal paired-dominating sets of distinct cardinalities. To show NP-completeness, we use exactly the same arguments as in the proof of theorem 3.

Given an instance $J = (C, X)$ of the 3Sat problem, construct the graph $G = G(J) \in A \cap B$ as in the proof of theorem 3. Let D be a minimal paired-dominating set of G. Since the cardinality $|D|$ is even, $i_{ev}(G) \geq n$, $\alpha_{ev}(G) = n + 1$ and $\gamma_{\mathrm{pr}}(G) \leq |D| \leq \Gamma_{\mathrm{pr}}(G)$, by using proposition 2 we obtain that $|D| \in \{2n, 2n + 2\}$. The graph G has always a minimal paired-dominating set of cardinality $2n + 2$, for example $\{c_1, c_2\} \cup \{y_i, z_i \mid i = 1, 2, \ldots, n\}$ is such a set. Moreover, there exists a satisfying truth assignment for C if and only if $i_{ev}(G) = n$. But since $\gamma_{\mathrm{pr}}(G) = 2i_{ev}(G) = 2n$, we also obtain that there exists a satisfying truth assignment for C if and only if $\gamma_{\mathrm{pr}}(G) = 2n$, i. e. if and only if the graph G has a minimal paired-dominating set of cardinality $2n$. Theorem 10 is proved.

Conclusions

In this paper, we prove that the problem of recognising well *ve*-covered graphs is co-NP-complete. We show that the co-NP-completeness result holds for $K_{1,4}$-free graphs as well. Also, we prove that the problem of recognising well *ev*-covered graphs is still co-NP-complete for some classes defined by certain forbidden induced subgraphs such as $\{C_4, \overline{K_1 \cup C_4}, K_{1,5}, 2P_4\}$-free graphs. We introduce new hereditary classes of graphs, the perfectly well *ve*-covered graphs and the perfectly well *ev*-covered graphs, and we characterise them by finite sets of forbidden induced subgraphs, thus obtaining new polynomial-time recognisable classes, where parameters $i_{ve}(G)$, $\alpha_{ve}(G)$ and $i_{ev}(G)$, $\alpha_{ev}(G)$ are easy to compute. We prove that the problem of deciding «$i_{ve}(G) \leq k$» («$i_{ev}(G) \leq k$», respectively) is NP-complete even if the graph has minimal independent *ve*-dominating (minimal independent *ev*-dominating, respectively) sets of at most two cardinalities differing by one. Finally, we introduce a new class of graphs, the well paired-dominated graphs, and we show that recognising a well paired-dominated graph is a co-NP-complete problem.

An interesting open problem is to investigate the computational complexity of the Non-Well *ve*-Covered Graph problem (Non-Well *ev*-Covered Graph problem, respectively) for $K_{1,3}$-free graphs ($K_{1,t}$-free graphs, where $t \in \{3, 4\}$, respectively). Note that our approach does not apply to this case.

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