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О КЛАССИФИКАЦИИ С ТОЧНОСТЬЮ ДО ИЗОТОПИИ ПРОСТЫХ 3-МЕРНЫХ КОММУТАТИВНЫХ АЛГЕБР НИЛЬ-РАНГА 1

В. И. ГЛИЗБУРГ¹⁾, С. В. ПЧЕЛИНЦЕВ²⁾

¹⁾Московский городской педагогический университет,

2-й Сельскохозяйственный проезд, 4, корп. 1, 129226, г. Москва, Россия

²⁾Финансовый университет при Правительстве Российской Федерации,
пр. Ленинградский, 49/2, 125167, г. Москва, Россия

Аннотация. Доказано, что любые две простые 3-мерные унитарные коммутативные алгебры ниль-ранга 1 изотопны, если основное поле является алгебраически замкнутым и имеет характеристику, отличную от 2 и 3. Таким образом, завершена классификация с точностью до изотопии простых 3-мерных унитарных коммутативных алгебр, содержащих ниль-элементы индекса 2.

Ключевые слова: простая 3-мерная алгебра; коммутативная алгебра; ниль-элемент; ниль-ранг; изотоп.

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Авторы:

Вита Иммануиловна Глизбург – кандидат физико-математических наук, доктор педагогических наук, доцент; профессор департамента методики обучения Института педагогики и психологии образования.

Сергей Валентинович Пчелинцев – доктор физико-математических наук, профессор; профессор кафедры математики и анализа данных факультета информационных технологий и анализа больших данных.

Authors:

Vita I. Glizburg, PhD (physics and mathematics), doctor of science (pedagogy), docent; professor at the department of teaching methods, Institute of Pedagogy and Psychology of Education. glizburg@mail.ru

<https://orcid.org/0000-0003-1230-6931>

Sergey V. Pchelintsev, doctor of science (physics and mathematics), full professor; professor at the department of mathematics and data analysis, faculty of information technology and big data analysis.

pchelintsev@mail.ru

<https://orcid.org/0000-0001-7857-9532>

ON CLASSIFICATION UP TO ISOTOPY OF SIMPLE 3-DIMENSIONAL COMMUTATIVE ALGEBRAS OF NIL-RANK 1

V. I. GLIZBURG^a, S. V. PCHELINTSEV^b

^aMoscow City University, 4 Vtoroj Selskhozjajstvennyj proezd, 1 building, Moscow 129226, Russia

^bFinancial University under the Government of the Russian Federation,
49/2 Leningradskij Avenue, Moscow 125167, Russia

Corresponding author: V. I. Glizburg (glizburg@mail.ru)

Abstract. It is proved that any two simple 3-dimensional unital commutative algebras of nil-rank 1 are isotopic if the ground field is algebraically closed and has characteristic different from 2 and 3. Thus, the classification up to isotopy of simple 3-dimensional unital commutative algebras containing nil-elements of index 2 is completed.

Keywords: simple 3-dimensional algebra; commutative algebra; nil-element; nil-rank; isotope.

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Introduction

Everywhere below it is assumed that the main field is algebraically closed and its characteristic is different from 2 and 3.

An algebra with unit 1 (neutral element under multiplication) is called unital. An algebra is called simple if it has the non-zero multiplication and does not contain its own ideals. Algebra is called isotopically simple if every its isotope is a simple algebra.

The interest in the study of isotopically simple algebras is associated with Bruck's theorem on composition series [1]: every finite-dimensional algebra A has the isotope B , in which there is a composition series

$$B = B_0 \triangleright B_1 \triangleright B_2 \triangleright \dots \triangleright B_k = 0,$$

where $B_{i+1} \triangleleft B_i$ (ideal) and factor algebras B_i / B_{i+1} , $i = 0, k-1$, are either isotopically simple or 1-dimensional algebras with zero multiplication.

In the same work [1] R. H. Bruck proved that there are no 2-dimensional isotopically simple algebras.

Studying of 3-dimensional isotopically simple algebras is quite a difficult task.

It is known [2] that every simple 3-dimensional anticommutative algebra is isotopic to Lie algebra sl_2 second-order traceless matrices.

Work [2] provides a fairly complete list of papers related to the study of isotopes.

A significant number of publications are devoted to the study of isotopes associated with automorphisms of commutative associative algebras (see [3]).

Papers [4–6] studied isotopes of alternative, $(-1, 1)$ and Jordan algebras. In work [7] it was shown that among the isotopes of Jordan algebra of a symmetric bilinear non-degenerate form on an infinite-dimensional vector space there are both simple algebras and algebras containing ideals with zero multiplication.

A non-zero element a is called a nil-element of index 2 if $a^2 = 0$. The maximum number of linearly independent nil-elements of index 2 is called the nil-rank of the algebra. In publication [8] is given a classification up to isotopy of simple 3-dimensional unital commutative algebras of nil-rank 2. It turns out that every such algebra is isotopic to Jordan algebra J of a symmetric bilinear non-degenerate form on 2-dimensional vector space.

This work is devoted to proving an analogue of the above result for algebras of nil-rank 1. Let's describe its contents.

Let C be a simple 3-dimensional unital commutative algebra of nil-rank 1 and $a \in C$ be a nil-element of index 2. Then the operator R_a is either nilpotent or there exists $0 \neq b \in C$ such that $ab = b$. In the first case we say that C is of Engel type, and in the second case that it is regular.

First we prove that an algebra of Engel type is isotopic to some regular algebra. Then we classify up to isomorphism the regular algebras and define the conditions under which a regular algebra has nil-rank equal to 1.

The work considers the commutative algebra, which we called special,

$$D = \langle 1, x, y \mid x^2 = 0, xy = y, y^2 = 1 \rangle,$$

and the main result is proven in the work: every simple 3-dimensional unital commutative algebra of nil-rank 1 is isotopic to the special algebra D .

It can be proved that all unital commutative isotopes of Jordan algebra J and special algebra D are standard, and, therefore, they have the same nil-rank as the original algebra.

Thus, taking into account the publication [8] and the made remark, the classification is obtained accurate to isotopy of the simple 3-dimensional unital commutative algebras, containing nil-elements of index 2.

Every such algebra is isotopic to one and only one of the following three algebras depending on its nil-rank r : cyclotomic algebra for $r = 3$ ([2], theorem 5), Jordan algebra of non-degenerate bilinear symmetric form for $r = 2$ ([8], theorem 1), or special algebra for $r = 1$ (see lemma 6 and theorems 2 and 3).

Albert isotopes

In 1942 A. A. Albert [9] introduced the concept of isotope. Let's remember that the linear algebras $(A, \cdot)$ and $(B, \circ)$ over the main field F are called isotopic, if there is a triple $\Lambda = (\varphi, \psi, \xi)$ of linear isomorphisms $A \rightarrow B$, satisfying the condition $x^\varphi \circ y^\psi = (x \cdot y)^\xi$ for any $x, y \in A$. In this case, the triple of isomorphisms Λ is called isotopy. The isotopy relation is a broader equivalence relation than the isomorphism relation.

Let $f, g: A \rightarrow A$ be the linear automorphisms of the additive structure of algebra $(A, \cdot)$. Let us define the algebra $A^* = A^{(f, g)}$ on the same linear space A with respect to the new multiplication $x * y = x^f y^g$ for any $x, y \in A$. Algebra A^* is called the main isotope of algebra A .

It is known (see [4] or [9]) that two algebras are isotopic if and only if one of them is isomorphic to the main isotope of the other. In particular, if the algebras A and B are unital and isotopic, then the main isotope B^* is also unital.

Albert theorem. *If isotope $A^* = A^{(\varphi, \psi)}$ is unital, then there are elements $g, h \in A$ such that $\varphi R_g = \psi L_h = 1_A$.*

A short proof of this theorem can be found in paper [4].

Definition 1. An element a of the algebra A is called R -invertible, if the operator $R_a: x \mapsto xa$ of right multiplication by the element a is invertible (non-degenerate). Note that the inverse operator R_a^{-1} , generally speaking, is not a right multiplication operator (see, for example, [8]).

Let's denote the set of non-zero scalars from F as $F^\times$. The following two lemmas, proven in publication [8], are valid.

Lemma 1. *Let A be an algebra, $\sigma, \tau \in F^\times$ and $1 = 1_A$. Then the isotope $A^* = A^{(\sigma 1, \tau 1)}$ with respect to the operation $a * b = (\sigma a)(\tau b)$ and the algebra A are isomorphic.*

Definition 2. An isotope of the algebra C of the form $C^* = C^{(\sigma\varphi, \varphi)}$, $\sigma \in F^\times$ is called standard.

Lemma 2. *The algebra C and its standard isotope $C^* = C^{(\sigma\varphi, \varphi)}$ have equal nil-ranks.*

Simple algebras of Engel type

Throughout below, unless otherwise is stated, the term «algebra» means that we are talking about the 3-dimensional unital commutative algebra C over an algebraically closed field F of the characteristic different from 2 and 3.

Definition 3. An element $a \neq 0$ of the algebra C is called nil-element of index 2 if $a^2 = 0$.

Definition 4. The algebra C is called the algebra of Engel type, if for some nil-element a of index 2, the operator R_a is nilpotent.

If C is a simple algebra of Engel type and its nil-rank is 1, then the nilpotency of the operator R_b holds for any nil-element b of index 2, since a and b are proportional.

Lemma 3. *In any simple algebra C of Engel type it is possible to choose the basis $(1, x, y)$ so that $x^2 = 0$, $xy = 1$. Moreover, the operator R_x is nilpotent of index 3.*

Proof. Let's note that if $(1, x, y)$ is the basis and $x^2 = 0$, $xy = 1$, then

$$1R_x^2 = 0, xR_x = 0, yR_x^2 = x, yR_x^3 = 0.$$

Let a be a nil-element of index 2. Then the elements $1, a$ are linearly independent.

Let's consider the subspace $U = \langle 1, a \rangle$, generated by elements $1, a$. It is invariant under R_a , which means that on the one-dimensional space C/U the nilpotent operator R_a has zero effect. Then there is $b \notin U$ such that $ba = \lambda 1 + \mu a$ for some $\lambda, \mu \in F$.

Note that the elements $(1, a, b)$ form the basis of the algebra C . If $\lambda = 0$, then $ba = \mu a$ and $\langle a \rangle$ is a proper ideal in algebra, which contradicts the simplicity of the algebra C . This means $\lambda \neq 0$. Therefore, $(b - \mu 1)a = \lambda 1$

and $(b - \mu 1)(\lambda^{-1}a) = 1$. Then after the replacement $x := \lambda^{-1}a$, $y := b - \mu 1$ we obtain a basis of the algebra C of the required form. The lemma is proven.

Let's note that by virtue of lemma 3 it is necessary to know the expansion of the vector y^2 in the basis $(1, x, y)$ to describe the structure of the algebra C . Let

$$C = \langle 1, x, y \mid x^2 = 0, xy = yx = 1, y^2 = \tau_1 1 + \tau_2 x + \tau_3 y \rangle.$$

Lemma 4. *Under the conditions of lemma 3 in the algebra C we can choose the basis $(1, x, y)$ so that $x^2 = 0$, $xy = 1$ and one of the conditions is completed:*

(i) $y^2 = x$;

(ii) $y^2 = \tau x + y$, where $\tau \in F$.

Proof. According to lemma 3 in the algebra C we can choose the basis $(1, x, y)$ such that $x^2 = 0$, $xy = 1$. Let $y' := y + \rho x$. Then

$$\begin{aligned} xy' &= xy + \rho x^2 = xy = 1, \\ y'^2 &= (y + \rho x)^2 = y^2 + 2\rho xy = (2\rho + \tau_1)1 + \tau_2 x + \tau_3 y. \end{aligned}$$

So, choosing ρ so that $2\rho + \tau_1 = 0$, we can assume that the basic elements $1, x, y$ satisfy the following equalities:

$$x^2 = 0, xy = 1, y^2 = \tau_2 x + \tau_3 y.$$

Let's make one more replacement: $x' := \mu x$, $y' := \mu^{-1}y$. Then

$$x'^2 = 0, x'y' = 1, \mu^2 y'^2 = y^2 = \tau_2 x + \tau_3 y = \tau_2 \mu^{-1} x' + \tau_3 \mu y'.$$

Multiplying both sides of the last equality by μ^{-2} , we get

$$y'^2 = \tau_2 \mu^{-3} x' + \tau_3 \mu^{-1} y'.$$

Let's consider two cases:

(i) $\tau_3 = 0$. Then $y'^2 = \tau_2 \mu^{-3} x'$ and, putting $\mu^3 = \tau_2$, we get $y'^2 = x'$;

(ii) $\tau_3 \neq 0$. Then, putting $\mu = \tau_3$, $\tau = \tau_2 \mu^{-3}$, we have $y'^2 = \tau x' + y'$.

Reduction to regular algebras

Definition 5. The simple algebra C is called regular if it is possible to choose the canonical basis $(1, a, b)$ such that $a^2 = 0$, $ab = b$.

Remark. Let C be a regular algebra with canonical basis $(1, a, b)$. Then the operator R_a is not nilpotent. Conversely, if $a^2 = 0$ and the operator R_a is not nilpotent, then it has an eigenvector b corresponding to a non-zero eigenvalue λ , that is $ab = \lambda b$. It is clear that $b \notin \langle 1, a \rangle$. Replacing the vector a by the vector $a' = \lambda^{-1}a$, we obtain the canonical basis $(1, a', b)$.

Let's now prove the following result.

Theorem 1. *A simple algebra of Engel type has a regular isotope.*

Proof. Let's consider two cases of algebras of Engel type indicated in lemma 4.

(i) Let in the algebra C the following relations be satisfied for the elements of the canonical basis $(1, x, y)$:

$$x^2 = 0, xy = 1, y^2 = x.$$

Let us set $c := 1 + y$ and show that the operator R_c is invertible:

$$1c = c = 1 + y, xc = x(1 + y) = 1 + x, yc = y(1 + y) = x + y.$$

This means, that by identifying, as usual, the operator and its matrix, we have

$$R_c = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \det(R_c) = 2 \neq 0.$$

Let's put

$$\varphi := R_c^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Let's consider the elements

$$a := 1 + x, \quad b := x + y, \quad e = c^2 = 1 + x + 2y.$$

Then we have

$$a^\varphi = (1, 1, 0) \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = (0, 1, 0) = x,$$

$$b^\varphi = (0, 1, 1) \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = (0, 0, 1) = y.$$

Let's consider the isotope $C^* = C^{(\varphi, \varphi)}$, where $u * v = u^\varphi v^\varphi$, and the products in it

$$e * v = cv^\varphi = v(\varphi R_c) = v, \quad a * a = a^\varphi a^\varphi = xx = 0,$$

$$a * b = a^\varphi b^\varphi = xy = 1 = \frac{1}{2}e + \frac{1}{2}a - b, \quad b * b = b^\varphi b^\varphi = y^2 = x = a - 1.$$

Let's put $a' = -a$, $b' = b - \frac{1}{2}e$. We have $a' * a' = a * a = 0$ and

$$a' * b' = -a * \left(b - \frac{1}{2}e\right) = -\left(\frac{1}{2}e + \frac{1}{2}a - b\right) + \frac{1}{2}a = b - \frac{1}{2}e = b'.$$

It's clear that the operator $R_{a'}$ is not nilpotent.

(ii) Let the following relations be satisfied in algebra:

$$x^2 = 0, \quad xy = 1, \quad y^2 = \tau x + y.$$

Let us assume that $\tau + 2 \neq 0$. Let's put $c := 1 + y$ and show that the operator R_c is invertible:

$$1c = c = 1 + y, \quad xc = 1 + x, \quad yc = y(1 + y) = \tau x + 2y.$$

It means that,

$$R_c = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \tau & 1 \end{pmatrix}, \quad \det(R_c) = \tau + 2 \neq 0.$$

Let's put

$$\varphi := R_c^{-1} = \frac{1}{\tau + 2} \begin{pmatrix} 2 & \tau & -1 \\ -2 & 2 & 1 \\ \tau & -\tau & 1 \end{pmatrix},$$

$$a := 1 + x, \quad b := \tau x + 2y, \quad e := c^2 = 1 + \tau x + 3y.$$

Then

$$a^\varphi = (1, 1, 0) \frac{1}{\tau + 2} \begin{pmatrix} 2 & \tau & -1 \\ -2 & 2 & 1 \\ \tau & -\tau & 1 \end{pmatrix} = (0, 1, 0) = x,$$

$$b^\varphi = (0, \tau, 2) \frac{1}{\tau + 2} \begin{pmatrix} 2 & \tau & -1 \\ -2 & 2 & 1 \\ \tau & -\tau & 1 \end{pmatrix} = (0, 0, 1) = y.$$

Let's consider the isotope $C^* = C^{(\varphi, \varphi)}$, where $u * v = u^\varphi v^\varphi$.

We have

$$e * v = cv^\varphi = v(\varphi R_c) = v, \quad a * a = a^\varphi a^\varphi = xx = 0,$$

$$a * b = a^\varphi b^\varphi = xy = 1 = \frac{1}{\tau + 2}e + \frac{\tau}{\tau + 2}a - \frac{3}{\tau + 2}b.$$

The operator R_a^* of right multiplication by the element a in the algebra C^* is not nilpotent, since the vector $e + \frac{(3\tau-1)(\tau+2)}{3}a - 3b$ is an eigenvector of R_a , corresponding to the eigenvalue $\lambda = -\frac{3}{\tau+2}$:

$$\begin{aligned} a * \left(e + \frac{(3\tau-1)(\tau+2)}{3}a - 3b \right) + \frac{3}{\tau+2} \left(e + \frac{(3\tau-1)(\tau+2)}{3}a - 3b \right) &= \\ &= a - 3a * b + \frac{3}{\tau+2} \left(e + \frac{(3\tau-1)(\tau+2)}{3}a - 3b \right) = \\ &= a - 3 \left(\frac{1}{\tau+2} + \frac{\tau}{\tau+2}a - \frac{3}{\tau+2}b \right) + \frac{3}{\tau+2} \left(e + \frac{(3\tau-1)(\tau+2)}{3}a - 3b \right) = 0. \end{aligned}$$

Let now $\tau = -2$ and the following relations hold in the algebra:

$$x^2 = 0, \quad xy = 1, \quad y^2 = -2x + y.$$

Let's put $c := 1 - y$ and show that the operator R_c is invertible:

$$1c = c = 1 - y, \quad xc = x(1 - y) = -1 + x, \quad yc = y(1 - y) = 2x.$$

Then

$$R_c = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \det(R_c) = 2 \neq 0.$$

Let's put

$$\varphi := R_c^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ -2 & -2 & 1 \end{pmatrix},$$

$$a := -1 + x, \quad b := 2x, \quad e := c^2 = 1 - 2x - y.$$

Then

$$a^\varphi = (-1, 1, 0) \cdot \frac{1}{2} \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ -2 & -2 & 1 \end{pmatrix} = (0, 1, 0) = x,$$

$$b^\varphi = (0, 2, 0) \cdot \frac{1}{2} \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ -2 & -2 & 1 \end{pmatrix} = (0, 0, 1) = y.$$

Let's consider the isotope $C^* = C^{(\varphi, \varphi)}$, where $u * v = u^\varphi v^\varphi$. We have

$$e * v = cv^\varphi = v(\varphi R_c) = v, \quad a * a = a^\varphi a^\varphi = xx = 0,$$

$$a * b = a^\varphi b^\varphi = xy = 1 = -a + \frac{1}{2}b.$$

The operator R_a^* of right multiplication by the element a in the algebra C^* is not nilpotent, since it has an eigenvector $2a - b$, corresponding to the eigenvalue $\lambda = \frac{1}{2}$:

$$a * (2a - b) - \frac{1}{2}(2a - b) = -a * b - a + \frac{1}{2}b = -\left(-a + \frac{1}{2}b\right) - a + \frac{1}{2}b = 0.$$

The theorem is proven.

Classification of regular algebras up to isomorphism

Special algebra D . Let us define a special algebra.

Definition 6. Let's call a special the commutative algebra of the form

$$D = \langle 1, x, y \mid x^2 = 0, xy = y, y^2 = 1 \rangle.$$

Lemma 5. *The special algebra D is regular.*

Proof. Let $a = \alpha 1 + \beta x + \gamma y$ be a nil-element of index 2 in the algebra D . Then

$$0 = (\alpha 1 + \beta x + \gamma y)^2 = (\alpha^2 + \gamma^2)1 + 2\alpha\beta x + 2(\alpha + \beta)\gamma y.$$

Since the elements $1, x, y$ are linearly independent, then

$$\alpha^2 + \gamma^2 = 0, \alpha\beta = 0, (\alpha + \beta)\gamma = 0.$$

Let's prove that $\alpha = 0$. Let, by contradiction, $\alpha \neq 0$. Then $\beta = 0$ and $\gamma = 0$, which implies $\alpha = 0$.

So, $\alpha = 0$, but then $\gamma = 0$, which means that the elements a and x are proportional, that is the nil-rank of the algebra D is 1.

Further, the operator R_x is not nilpotent, since y is its eigenvector, corresponding to eigenvalue 1.

Let us show now that the algebra D is simple. Let I be a proper ideal in D .

1. Let's first prove that I cannot contain any of the basis vectors $1, x, y$. Obviously, $1 \notin I$. If $y \in I$, then $1 = y^2 \in I$. If $x \in I$, then $y = xy \in I$.

2. Let's prove now that I cannot contain a vector of the form $\alpha 1 + \beta x + \gamma y$, where exactly two of the three scalars α, β, γ are non-zero. If $x + \gamma y \in I$ and $\gamma \neq 0$, then $\gamma y = x(x + \gamma y) \in I$, which is impossible due to the item 1. If $1 + \beta x \in I$, then $x = x(1 + \beta x) \in I$, which is also impossible. If $1 + \gamma y \in I$, then $x + \gamma y = x(1 + \gamma y) \in I$, which is impossible due to the previous one.

3. Let's prove that I cannot contain a vector of the form $u = 1 + \beta x + \gamma y$ with non-zero scalars β, γ . If $u \in I$, then $I \ni ux = (1 + \beta x + \gamma y)x = x + \gamma y$, which contradicts the item 2.

The lemma is proven.

Basic types of regular algebras. Let's consider the unital commutative algebra

$$D(\rho_1, \rho_2, \rho_3) := \langle 1, a, b \mid a^2 = 0, ab = b, b^2 = \rho_1 1 + \rho_2 a + \rho_3 b \rangle,$$

where $\rho_i \in F$, $i = 1, 2, 3$ and $\rho_1 \neq 0$.

Note that every regular algebra has the indicated structure. The algebra $D(0, \rho_2, \rho_3)$ is not simple, since $\langle a, b \rangle$ is its ideal.

Lemma 6. *The regular algebra $D(\rho_1, \rho_2, \rho_3)$ is isomorphic to one of the algebras:*

$$D(\rho_1, \rho_2, 1), D(\rho_1, 1, 0), D(\rho_1, 0, 0).$$

Moreover, $D(\rho_1, 0, 0)$ for $\rho_1 \neq 0$ is isomorphic to the special algebra D .

Proof. Let $b' = \lambda b$, $\lambda \neq 0$. Then $ab' = b'$ and

$$\lambda^{-2}b'^2 = b^2 = \rho_1 1 + \rho_2 a + \rho_3 b = \rho_1 1 + \rho_2 a + \rho_3 \lambda^{-1}b'.$$

After multiplying both sides of this equality by λ^2 we get $b'^2 = \lambda^2 \rho_1 1 + \lambda^2 \rho_2 a + \lambda \rho_3 b'$.

If $\rho_3 \neq 0$, then for a suitable λ we have $\lambda \rho_3 = 1$, hence the algebra $D(\rho_1, \rho_2, \rho_3)$ is isomorphic to the algebra $D(\rho'_1, \rho'_2, 1)$ for some scalars ρ'_1, ρ'_2 .

If $\rho_3 = 0$, then in the algebra $D(\rho_1, \rho_2, 0)$ we have $b^2 = \rho_1 1 + \rho_2 a$. Let us assume that $\rho_2 \neq 0$. Then for $b' = \lambda b$, $\lambda \neq 0$, we have $ab' = b'$ and $b'^2 = \lambda^2 \rho_1 1 + \lambda^2 \rho_2 a$. Let's choose λ so that $\lambda^2 \rho_2 = 1$. Then the algebra $D(\rho_1, \rho_2, 0)$ is isomorphic to the algebra $D(\rho'_1, 1, 0)$ for some scalar ρ'_1 .

If $\rho_2 = \rho_3 = 0$, then the algebra $D(\rho_1, 0, 0)$ satisfies the following relations: $ab = b, b^2 = \rho_1 1$. Since $\rho_1 \neq 0$, then, similar to the previous one, the algebra $D(\rho_1, 0, 0)$ is isomorphic to the algebra $D(1, 0, 0)$, which coincides with the special algebra D . The lemma is proven.

Nil-ranks of the algebras $D(\rho_1, \rho_2, 1)$ and $D(\rho, 1, 0)$. Let's prove the following lemma.

Lemma 7. *The following statements are true:*

(i) *the nil-rank of the algebra $D(\rho, 1, 0)$ is equal to 1 if and only if $\rho \neq -\frac{1}{2}$;*

(ii) *the nil-rank of the algebra $D(\rho_1, \rho_2, 1)$ is equal to 1 if and only if $(2\rho_1 + \rho_2)^2 + \rho_1 \neq 0$.*

Proof. Let us assume that $z = \beta a + \gamma b$ is a nil-element of index 2 in the algebra $D(\rho_1, \rho_2, \rho_3)$, where $\rho_1 \neq 0$. Then $0 = z = (\beta a + \gamma b)^2 = 2\beta\gamma b + \gamma^2(\rho_1 1 + \rho_2 a + \rho_3 b)$.

Since the elements $1, a, b$ are linearly independent, then $\gamma^2\rho_1 = 0$ and $\gamma = 0$, that is the elements z and a are proportional. So, without loss of generality, we can assume that the nil-element of index 2 has the form $z = 1 + \beta a + \gamma b$.

(i) In the algebra $D(\rho, 1, 0)$ we have

$$\begin{aligned} 0 = z^2 &= (1 + \beta a + \gamma b)^2 = 1 + \gamma^2 b^2 + 2\beta a + 2\gamma b + 2\beta\gamma ab = \\ &= (1 + \gamma^2\rho)1 + (2\beta + \gamma^2)a + 2\gamma(1 + \beta)b. \end{aligned}$$

Since the elements $1, a, b$ are linearly independent, then

$$1 + \gamma^2\rho = 0, \quad 2\beta + \gamma^2 = 0, \quad 2\gamma(1 + \beta) = 0.$$

From here we get $\gamma \neq 0$, $\beta = -1$, $2\beta\rho = 1$, which means $\rho = -\frac{1}{2}$.

(ii) In the algebra $D(\rho_1, \rho_2, 1)$ we have

$$\begin{aligned} 0 = z^2 &= (1 + \beta a + \gamma b)^2 = 1 + \gamma^2 b^2 + 2\beta a + 2\gamma b + 2\beta\gamma ab = \\ &= 1 + \gamma^2(\rho_1 + \rho_2 a + b) + 2\beta a + 2\gamma b + 2\beta\gamma ab = \\ &= (1 + \gamma^2\rho_1)1 + (2\beta + \gamma^2\rho_2)a + (2\beta + \gamma + 2)\gamma b. \end{aligned}$$

Note that $\gamma \neq 0$ and the following equalities are true:

$$1 + \gamma^2\rho_1 = 0, \quad 2\beta + \gamma^2\rho_2 = 0, \quad 2\beta + \gamma + 2 = 0.$$

From the first two equalities we obtain

$$0 = (1 + \gamma^2\rho_1)\rho_2 - (2\beta + \gamma^2\rho_2)\rho_1 = \rho_2 - 2\beta\rho_1.$$

Therefore, $\beta = \frac{\rho_2}{2\rho_1}$ and $\gamma = -2\beta - \frac{\rho_2}{\rho_1}$. Then the equality $1 + \gamma^2\rho_1 = 0$ takes the form $1 + \rho_1 \left(-\frac{\rho_2}{\rho_1} - 2 \right)^2 = 0$.

After multiplying both sides of this equality by ρ_1 we have $(2\rho_1 + \rho_2)^2 + \rho_1 \neq 0$. The lemma is proven.

Standard isotopes of the algebra D

Definition 7. Let $E(\alpha, \beta)$ denote the unital commutative algebra with the basis $(1, x, y)$ and the multiplication table

$$x^2 = 0, \quad xy = \sigma_1 1 + \sigma_2 x + \sigma_3 y, \quad y^2 = \tau_1 1 + \tau_2 x + \tau_3 y,$$

where

$$\begin{aligned} \sigma_1 &= -\beta\delta, \quad \sigma_2 = 2\alpha\beta\delta, \quad \sigma_3 = (\beta^2 + 1)\delta, \\ \tau_1 &= (\alpha + 1)\delta, \quad \tau_2 = -2\alpha(\alpha + 1)\delta, \quad \tau_3 = -2\beta\delta, \\ \delta &= (\alpha - \beta^2 + \alpha\beta^2 + 1)^{-1}. \end{aligned}$$

Lemma 8. *The algebra $E(\alpha, \beta)$ for any scalars α, β , satisfying the condition $\alpha - \beta^2 + \alpha\beta^2 + 1 \neq 0$, is isotopic to the special algebra D .*

Proof. Let $\alpha, \beta \in F$ and $\alpha - \beta^2 + \alpha\beta^2 + 1 \neq 0$. In the special algebra D , let's consider the element $c := 1 + \alpha x + \beta y$ and calculate the operator R_c :

$$1c = 1 + \alpha x + \beta y, \quad x' := xc = x + \beta y, \quad y' := yc = \beta 1 + (1 + \alpha)y,$$

$$R_c = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \beta \\ \beta & 0 & 1 + \alpha \end{pmatrix}, \quad \det(R_c) = \delta^{-1} = \alpha - \beta^2 + \alpha\beta^2 + 1.$$

Let's assume $e := c^2 = 1 + \beta^2 1 + 2\alpha x + 2\beta y + 2\alpha\beta xy = (1 + \beta^2, 2\alpha, 2(\alpha + 1)\beta)$.

Since $\begin{vmatrix} 1 + \beta^2 & 2\alpha & 2(\alpha + 1)\beta \\ 0 & 1 & \beta \\ \beta & 0 & \alpha + 1 \end{vmatrix} \delta = 1$, then e, x', y' are linearly independent.

Let's put $\varphi := R_c^{-1} = \delta \begin{pmatrix} \alpha + 1 & -\alpha^2 - \alpha & \alpha\beta - \beta \\ \beta^2 & -\beta^2 + \alpha + 1 & -\beta \\ -\beta & \alpha\beta & 1 \end{pmatrix}$. Note that $x'^\varphi = x, y'^\varphi = y$. So (e, x', y') is the basis

of the isotope $D^* = D^{(\varphi, \varphi)}$, $u * v = u^\varphi x' v^\varphi$. We have

$$e * v = v, \quad x' * x' = 0, \quad x' * y' = xy = y = -\beta\delta e + 2\beta\delta x' + (\beta^2 + 1)\delta y',$$

$$y' * y' = y^2 = 1 = (\alpha + 1)\delta e - 2\alpha(\alpha + 1)\delta x' - 2\delta y',$$

the last two equalities are checked directly.

Thus, it is proved that the algebra D^* is isomorphic to the algebra $E(\alpha, \beta)$. The lemma is proven.

Main results

Isotopes of the algebra $D(\rho, 1, 0)$. Let's prove the following theorem.

Theorem 2. Algebra of the type $D(\rho, 1, 0)$ for $\rho \neq -\frac{1}{2}$ is isotopic to the algebra D .

Proof. By lemma 8 it is sufficient to understand that in the algebra $E(\alpha, 0)$ there are elements such that the equalities are satisfied

$$a^2 = 0, \quad ab = b, \quad b^2 = \rho 1 + a, \tag{1}$$

provided that $\rho \neq -\frac{1}{2}$. Note that for $\beta = 0$ we have the following parameter values, participating in the definition of the algebra $E(\alpha, 0)$:

$$\sigma_1 = \sigma_2 = \tau_3 = 0, \quad \sigma_3 = \delta = \frac{1}{1 + \alpha}, \quad \tau_1 = 1, \quad \tau_2 = -2\alpha.$$

Multiplication in the algebra $E(\alpha, 0)$ is given by the table

$$x^2 = 0, \quad xy = \frac{1}{1 + \alpha}y, \quad y^2 = 1 - 2\alpha x.$$

We will look for elements $a, b \in E(\alpha, 0)$ that satisfy relations (1) in the form

$$a = (\alpha + 1)x, \quad b = \theta y.$$

We have $0 = ab - b = (\alpha + 1)\theta xy - \theta y = (\alpha + 1)\theta \frac{1}{1 + \alpha}y - \theta y = 0$,

$$\begin{aligned} 0 &= b^2 - (\rho 1 + a) = \theta^2 y^2 - (\rho 1 + (\alpha + 1)x) = \\ &= \theta^2 (1 - 2\alpha x) - (\rho 1 + (\alpha + 1)x) = (\theta^2 - \rho)1 - (\alpha + 1 + 2\theta^2 \alpha)x. \end{aligned}$$

So, in order relations (1) to be satisfied, it is necessary and sufficient to fulfill the equalities $\theta^2 = \rho$, $\alpha(1+2\theta^2)+1=0$ or $\theta^2 = \rho$, $\alpha(1+2\rho)+1=0$. Since $\rho \neq -\frac{1}{2}$, then the last system is solvable with respect to the unknowns α and θ . The theorem is proven.

By virtue of lemmas 2 and 7 (i) the restriction $\rho \neq -\frac{1}{2}$ in theorem 2 is essential.

Isotopes of the algebra $D(\rho_1, \rho_2, 1)$. Let's prove the following theorem.

Theorem 3. *Algebra of the type $D(\rho_1, \rho_2, 1)$ for $(2\rho_1 + \rho_2)^2 + \rho_1 \neq 0$ is isotopic to the algebra D .*

Proof. By virtue of lemma 8, the algebra D is isotopic to the algebra $E(\alpha, \beta)$. Let's prove that the algebra $D(\rho_1, \rho_2, 1)$ is isomorphic to the algebra $E(\alpha, \beta)$ for some α, β , and $\beta^2 \neq -1$. Recall that the commutative algebra $E(\alpha, \beta)$ has the basis $(1, x, y)$ and the multiplication table

$$x^2 = 0, \quad xy = \sigma_1 1 + \sigma_2 x + \sigma_3 y, \quad y^2 = \tau_1 1 + \tau_2 x + \tau_3 y,$$

where

$$\begin{aligned} \sigma_1 &= -\beta\delta, \quad \sigma_2 = 2\alpha\beta\delta, \quad \sigma_3 = \gamma^{-1}\delta, \\ \tau_1 &= (\alpha+1)\delta, \quad \tau_2 = -2\alpha(\alpha+1)\delta, \quad \tau_3 = -2\beta\delta, \\ \gamma &= (\beta^2+1)^{-1}, \quad \delta = (\alpha - \beta^2 + \alpha\beta^2 + 1)^{-1}. \end{aligned}$$

It is enough to understand that in the algebra $E(\alpha, \beta)$ there are elements a, b , which

$$a^2 = 0, \quad ab = b, \quad b^2 = \rho_1 1 + \rho_2 a + b. \quad (2)$$

The operator R_x in the algebra $E(\alpha, \beta)$ has the form $R_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$. Its characteristic polynomial is

equal to $t^2(t - \sigma_3)$, this means that its Jordan form is a matrix of the form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$.

Let's check that the Jordan basis of the operator R_x consists of the vectors $1, x, b$, where

$$b := \theta_1 1 + \theta_2 x + y, \quad \theta_1 = \frac{\sigma_1}{\sigma_3}, \quad \theta_2 = \frac{\sigma_1 + \sigma_2 \sigma_3}{\sigma_3^2}.$$

Note that $\sigma_3 \neq 0$ and check the equality $bx = \sigma_3 b$:

$$\begin{aligned} bx &= (\theta_1 1 + \theta_2 x + y)x = \theta_1 x + xy = \theta_1 x + \sigma_1 1 + \sigma_2 x + \sigma_3 y = \\ &= \sigma_1 1 + (\theta_1 + \sigma_2)x + \sigma_3 y = \sigma_3 \left(\frac{\sigma_1}{\sigma_3} + \frac{\sigma_1 + \sigma_2 \sigma_3}{\sigma_3^2} x + y \right) = \\ &= \sigma_3 \left(\frac{\sigma_1}{\sigma_3} + \frac{\sigma_1 + \sigma_2 \sigma_3}{\sigma_3^2} x + y \right) = \sigma_3 (\theta_1 1 + \theta_2 x + y) = \sigma_3 b. \end{aligned}$$

Vectors a, b of the form $a := \frac{1}{\sigma_3} x$, $b = \theta_1 1 + \theta_2 x + y$ satisfy the first two relations (2). The validity of the first equality $a^2 = 0$ is obvious. Let's check the second equality $ab = b$:

$$ab = \frac{1}{\sigma_3} x \cdot b = \frac{1}{\sigma_3} \cdot bx = \frac{1}{\sigma_3} \cdot \sigma_3 b = b.$$

Note that the following equalities are true:

$$x = \sigma_3 a, \quad y = -\theta_1 1 - \theta_2 \sigma_3 a + b. \quad (3)$$

It remains to understand that according to the given scalars ρ_1, ρ_2 it is possible to choose α, β so that the third equality from relations (2) is satisfied. We have

$$\begin{aligned} b^2 &= (\theta_1 1 + \theta_2 x + y)^2 = \theta_1^2 + 2\theta_1\theta_2 x + 2\theta_1 y + 2\theta_2 xy + y^2 = \\ &= \theta_1^2 + 2\theta_1\theta_2 x + 2\theta_1 y + 2\theta_2 (\sigma_1 1 + \sigma_2 x + \sigma_3 y) + (\tau_1 1 + \tau_2 x + \tau_3 y) = \\ &= (\theta_1^2 + 2\theta_2\sigma_1 + \tau_1)1 + (2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2)x + (2\theta_1 + 2\theta_2\sigma_3 + \tau_3)y. \end{aligned}$$

Let's put

$$\xi := 2\theta_1 + 2\theta_2\sigma_3 + \tau_3. \quad (4)$$

Later we will show that for a suitable choice of scalars α, β the relation $\xi \neq 0$ holds. Continuing the calculation of b^2 , due to equalities (3) we have

$$\begin{aligned} b^2 &= (\theta_1^2 + 2\theta_2\sigma_1 + \tau_1)1 + (2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2)\sigma_3 a + \xi y = \\ &= (\theta_1^2 + 2\theta_2\sigma_1 + \tau_1 - \xi\theta_1)1 + (2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2 - \xi\theta_2)\sigma_3 a + \xi b, \end{aligned}$$

that is

$$b^2 = \lambda_1 1 + \lambda_2 a + \xi b, \quad (5)$$

where

$$\lambda_1 = \theta_1^2 + 2\theta_2\sigma_1 + \tau_1 - \xi\theta_1, \quad \lambda_2 = (2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2 - \xi\theta_2)\sigma_3.$$

Setting $b = \xi b'$, due to relation (5) we have $\xi^2 b'^2 = \lambda_1 1 + \lambda_2 a + \xi^2 b'$, hence,

$$a^2 = 0, \quad ab' = b', \quad b'^2 = \lambda_1 \xi^{-2} 1 + \lambda_2 \xi^{-2} a + b'.$$

To prove the isomorphism of the algebras $D(\rho_1, \rho_2, 1)$ and $E(\alpha, \beta)$, it is sufficient to establish the validity of the equalities

$$\rho_1 = \lambda_1 \xi^{-2}, \quad \rho_2 = \lambda_2 \xi^{-2}. \quad (6)$$

Let us prove that it is possible to select the scalars (α, β) so that equalities (6) are satisfied. Let us present the corresponding reasoning in the form of a sequence of points.

1. Let us show that $\xi = 2(\beta^2 - 3)\beta\gamma\delta$. Consistently we have

$$\begin{aligned} \theta_1 &= \frac{\sigma_1}{\sigma_3} = \sigma_1 \sigma_3^{-1} = -\beta\delta \cdot \gamma\delta^{-1} = -\beta\gamma, \\ \frac{\sigma_2}{\sigma_3} &= 2\alpha\beta\delta \cdot \gamma\delta^{-1} = 2\alpha\beta\gamma, \\ \frac{\sigma_1\sigma_2}{\sigma_3} &= -\beta\delta \cdot 2\alpha\beta\delta \cdot \gamma\delta^{-1} = -2\alpha\beta^2\gamma\delta, \\ \theta_2 &= \frac{\sigma_1 + \sigma_2\sigma_3}{\sigma_3^2} = \frac{1}{\sigma_3} \cdot \frac{\sigma_1}{\sigma_3} + \frac{\sigma_2}{\sigma_3} = -\gamma\delta^{-1}\beta\gamma + 2\alpha\beta\gamma = -\beta\gamma^2\delta^{-1} + 2\alpha\beta\gamma, \\ \theta_2\sigma_3 &= \frac{\sigma_1}{\sigma_3} + \sigma_2 = 2\alpha\beta\delta - \beta\gamma. \end{aligned}$$

Let us now transform the following expression:

$$\begin{aligned} -4\delta^{-1} + 2(2\alpha - 1)\gamma^{-1} &= -4(\alpha - \beta^2 + \alpha\beta^2 + 1) + 2(2\alpha - 1)(\beta^2 + 1) = \\ &= -4\alpha + 4\beta^2 - 4\alpha\beta^2 - 4 + 4\alpha\beta^2 - 2\beta^2 + 4\alpha - 2 = 2(\beta^2 - 3). \end{aligned}$$

Due to formula (4) we have

$$\begin{aligned}\xi &= 2\theta_1 + 2\theta_2\sigma_3 + \tau_3 = -2\beta\gamma - 2\beta\gamma + 4\alpha\beta\delta - 2\beta\delta = \\ &= -4\beta\gamma + 2(2\alpha - 1)\beta\delta = (-4\delta^{-1}\gamma + 2(2\alpha - 1))\beta\delta = \\ &= (-4\delta^{-1} + 2(2\alpha - 1)\gamma^{-1})\beta\gamma\delta = 2(\beta^2 - 3)\beta\gamma\delta.\end{aligned}$$

2. Let us show that $\lambda_1 = \gamma^2$. Using the equalities $\frac{\sigma_1}{\sigma_3} = -\beta\gamma$ and $\frac{\sigma_1\sigma_2}{\sigma_3} = -2\alpha\beta^2\gamma\delta$, we have

$$\theta_2\sigma_1 = \left(\frac{1}{\sigma_3} \frac{\sigma_1}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\right)\sigma_1 = \left(\frac{\sigma_1}{\sigma_3}\right)^2 + \frac{\sigma_1\sigma_2}{\sigma_3} = (\beta\gamma)^2 - 2\alpha\beta^2\gamma\delta = \beta^2\gamma(\gamma - 2\alpha\delta).$$

Let us now transform the following expression:

$$\begin{aligned}-4\alpha\beta^2\gamma^{-1} + (\alpha + 1)\gamma^{-2} + 2(\beta^2 - 3)\beta^2 &= -4\alpha\beta^2(\beta^2 + 1) + (\alpha + 1)(\beta^2 + 1)^2 + 2(\beta^2 - 3)\beta^2 = \\ &= -4\alpha\beta^4 - 4\alpha\beta^2 + \alpha\beta^4 + 2\alpha\beta^2 + \alpha + \beta^4 + 2\beta^2 + 1 + 2\beta^4 - 6\beta^2 = \\ &= -3\alpha\beta^4 - \alpha\beta^2 + \alpha + 3\beta^4 - 4\beta^2 + 1 = -(3\beta^2 - 1)(\alpha - \beta^2 + \alpha\beta^2 + 1) = -(3\beta^2 - 1)\delta^{-1}.\end{aligned}$$

Completing the calculation of λ_1 , we obtain

$$\begin{aligned}\lambda_1 &= \theta_1^2 + 2\theta_2\sigma_1 + \tau_1 - \xi\theta_1 = (\beta\gamma)^2 + 2\beta^2\gamma(\gamma - 2\alpha\delta) + (\alpha + 1)\delta + 2(\beta^2 - 3)\beta^2\gamma^2\delta = \\ &= 3(\beta\gamma)^2 - 4\alpha\beta^2\gamma\delta + (\alpha + 1)\delta + 2(\beta^2 - 3)\beta^2\gamma^2\delta = \\ &= 3(\beta\gamma)^2 + (-4\alpha\beta^2\gamma^{-1} + (\alpha + 1)\gamma^{-2} + 2(\beta^2 - 3)\beta^2)\gamma^2\delta = 3(\beta\gamma)^2 - (3\beta^2 - 1)\gamma^2 = \gamma^2.\end{aligned}$$

3. Let us show that $\lambda_2 = -2(\alpha + 2\beta^2 + \alpha\beta^2)\gamma^2\delta$.

Let's first consider the expression $\chi := \lambda_2(\gamma^2\delta)^{-1}$:

$$\chi = \lambda_2(\gamma^2\delta)^{-1} = (2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2 - \xi\theta_2)\sigma_3(\gamma^2\delta)^{-1}.$$

Note that $\sigma_3(\gamma^2\delta)^{-1} = \gamma^{-1}\delta\gamma^{-2}\delta^{-1} = \gamma^{-3}$. Hence, $\chi\gamma^3 = 2\theta_1\theta_2 + 2\theta_2\sigma_2 + \tau_2 - \xi\theta_2$.

Because $\theta_1 = -\beta\gamma$, $\theta_2 = -\beta\gamma^2\delta^{-1} + 2\alpha\beta\gamma$, $\sigma_2 = 2\alpha\beta\delta$, then

$$\chi\gamma^3 = \delta^{-1}(8\alpha^2\beta^2\gamma\delta^2 - 8\alpha\beta^2\gamma^2\delta + 2\beta^2\gamma^3 + \xi\beta\gamma^2 + \tau_2\delta - 2\xi\alpha\beta\gamma\delta).$$

Let's note that

$$\chi = p + q + r,$$

where $p := \gamma^{-3}\delta^{-1}(8\alpha^2\beta^2\gamma\delta^2 - 8\alpha\beta^2\gamma^2\delta)$, $q := \gamma^{-3}\delta^{-1}(2\beta^2\gamma^3 + \xi\beta\gamma^2)$, $r := \gamma^{-3}\delta^{-1}(\tau_2\delta - 2\xi\alpha\beta\gamma\delta)$.

Let us calculate the values p , q , r , taking into account that

$$\gamma = (\beta^2 + 1)^{-1}, \quad \delta = (\alpha - \beta^2 + \alpha\beta^2 + 1)^{-1}.$$

Firstly, $p = 8\alpha\beta^2\gamma^{-2}(\alpha\delta - \gamma) = 8\alpha\beta^2(\beta^4 - 1)\delta$. Secondly, using $\xi = 2(\beta^2 - 3)\beta\gamma\delta$, we have

$$q = \beta\gamma^{-1}\delta^{-1}(2\beta\gamma + \xi) = 2\beta^2(\delta^{-1} + (\beta^2 - 3)) = 2\beta^2(\alpha\beta^2 + \alpha - 2).$$

Finally, thirdly, the following equalities are true:

$$\begin{aligned}r &= \gamma^{-3}(\tau_2 - 2\xi\alpha\beta\gamma) = -2\alpha\gamma^{-3}((\alpha + 1)\delta + \xi\beta\gamma) = \\ &= -2\alpha(\beta^2 + 1)(\alpha - 4\beta^2 + 3\beta^4 + 2\alpha\beta^2 + \alpha\beta^4 + 1)\delta.\end{aligned}$$

Then $\chi = p + q + r = -2(\alpha + 2\beta^2 + \alpha\beta^2)$, hence, $\lambda_2 = -2(\alpha + 2\beta^2 + \alpha\beta^2)\gamma^2\delta$, and item 3 is proven.

4. Using equalities (6) and items 1–3, we get the system

$$\begin{cases} \rho_1 = \lambda_1 \xi^{-2} = \gamma^2 \xi^{-2} = \frac{1}{4} \beta^{-2} (\beta^2 - 3)^{-2} \delta^{-2}, \\ \rho_2 = \lambda_2 \xi^{-2} = -\frac{1}{2} \beta^{-2} (\beta^2 - 3)^{-2} (\alpha + 2\beta^2 + \alpha\beta^2) \delta^{-1}, \end{cases} \quad (7)$$

where $\frac{\rho_2}{\rho_1 \delta} = -2(\alpha + 2\beta^2 + \alpha\beta^2)$. Assuming $\rho := \frac{\rho_2}{\rho_1}$, we have

$$\rho(\alpha - \beta^2 + \alpha\beta^2 + 1) = -2(\alpha + 2\beta^2 + \alpha\beta^2),$$

therefore, if $\rho + 2 \neq 0$, then

$$\alpha = f(\beta^2) := \frac{-4\beta^2 + \rho(\beta^2 - 1)}{(\beta^2 + 1)(\rho + 2)}. \quad (8)$$

If $\rho = -2$, then $1 - 3\beta^2 = 0$.

Let's note that system (7) is equivalent to the system

$$\begin{cases} \rho_1 = \frac{1}{4} \beta^{-2} (\beta^2 - 3)^{-2} \delta^{-2}, \\ \rho(\alpha - \beta^2 + \alpha\beta^2 + 1) = -2(\alpha + 2\beta^2 + \alpha\beta^2). \end{cases} \quad (9)$$

5. If $1 - 3\beta^2 \neq 0$, then $\rho + 2 \neq 0$ and equality (8) holds.

6. With the value $\alpha = f(\beta^2)$ specified in equality (8), we calculate δ^{-1} and ξ :

$$\delta^{-1}|_{\alpha=f(\beta^2)} = \frac{2}{\rho+2}(1-3\beta^2), \quad \xi|_{\alpha=f(\beta^2)} = (\rho+2) \frac{\beta(\beta^2-3)}{(1-3\beta^2)(\beta^2+1)}.$$

7. Parameter β must be chosen so that $k := (2\rho_1 + \rho_2)^2 + \rho_1 \neq 0$.

Let's assume that $k = 0$. Then $(2\rho_1 + \rho_2)^2 = -\rho_1$ and $2\rho_1 + \rho_2 = -i\sqrt{\rho_1}$. By virtue of equalities (6) and item 2 we have $2\gamma^2\xi^{-2} + \lambda_2\xi^{-2} = -i\gamma$, $2\gamma^2 + \lambda_2 + i\gamma\xi^2 = 0$. Using now the equalities $\lambda_2 = -2(\alpha + 2\beta^2 + \alpha\beta^2)\gamma^2\delta$ and $\xi = 2\beta(\beta^2 - 3)\gamma\delta$, proven in items 3 and 1, respectively, we get

$$-2\gamma^2(\delta(\alpha + 3i\beta + 2\beta^2 - i\beta^3 + \alpha\beta^2) - 1) = 0,$$

taking into account that $\gamma \neq 0$, we have

$$\alpha + 3i\beta + 2\beta^2 - i\beta^3 + \alpha\beta^2 = \delta^{-1}, \quad \alpha(1 + \beta^2) + 3i\beta + 2\beta^2 - i\beta^3 = \delta^{-1}.$$

By virtue of equality (8) and the representation δ^{-1} from item 6 we obtain

$$\begin{aligned} \frac{-4\beta^2 + \rho(\beta^2 - 1)}{\rho + 2} + 3i\beta + 2\beta^2 - i\beta^3 &= \frac{2}{\rho + 2}(1 - 3\beta^2), \\ -4\beta^2 + \rho(\beta^2 - 1) + (\rho + 2)(3i\beta + 2\beta^2 - i\beta^3) &= 2(1 - 3\beta^2), \\ (\rho + 2)(-i\beta^3 + 3\beta^2 + 3i\beta - 1) &= 0, \quad (\beta + i)^3 = 0. \end{aligned}$$

8. We must prove the existence of a solution to system (9) provided that $k \neq 0$. Let us choose β as the root of the equation

$$(1 - 3t^2)^2 = \rho_1(\rho + 2)^2 t^2 (t^2 - 3)^2,$$

where $\rho = \frac{\rho_2}{\rho_1}$, except numbers $0, \pm \frac{1}{\sqrt{3}}, \pm \sqrt{3}, \pm i$. Let's put $\alpha = \frac{-4\beta^2 + \rho(\beta^2 - 1)}{(\beta^2 + 1)(\rho + 2)}$. Then a pair of values (α, β)

is the solution of system (9) and $k \neq 0$, hence, the algebras $E(\alpha, \beta)$ and $D(\rho_1, \rho_2, 1)$ are isomorphic. The theorem is proven.

By virtue of lemmas 2 and 7 (ii) the restriction on parameters ρ_1, ρ_2 in theorem 3 is essential.

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