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О ПРОБЛЕМЕ ШЕМЕТКОВА ОПИСАНИЯ КРИТИЧЕСКИХ ФОРМАЦИЙ В КЛАССЕ τ-ЗАМКНУТЫХ σ-ЛОКАЛЬНЫХ ФОРМАЦИЙ

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Аннотация. Изучаются минимальные τ-замкнутые σ-локальные формации конечных групп, где σ – некоторое разбиение множества всех простых чисел $\mathbb{P}$, τ – произвольный подгрупповой функтор. Получено описание минимальных τ-замкнутых σ-локальных не $\mathfrak{H}$-формаций для произвольной σ-локальной формации классического типа, т. е. такой σ-локальной формации, которая имеет σ-локальное определение, все неабелевы значения которого являются σ-локальными формациями. Тем самым решена задача об описании критических формаций в классе τ-замкнутых σ-локальных формаций, предложенная Л. А. Шеметковым (1980).

Ключевые слова: конечная группа; подгрупповой функтор; τ-замкнутая формация; формационная σ-функция; σ-локальная формация; критическая σ-локальная формация; формация классического типа.

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ON SHEMETKOV'S PROBLEM OF DESCRIBING CRITICAL FORMATIONS IN THE CLASS OF τ-CLOSED σ-LOCAL FORMATIONS

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Abstract. The critical τ-closed σ-local formations of finite groups are studied, where σ is some partition of the set of all primes $\mathbb{P}$, τ is an arbitrary subgroup functor. A description of minimal τ-closed σ-local non-$\mathfrak{H}$-formations is obtained for an arbitrary σ-local formation of classical type, i. e. a σ-local formation that has a σ-local definition all of whose non-Abelian values are σ-local formations. Thus, the problem of describing critical formations in the class of τ-closed σ-local formations proposed by L. A. Shemetkov (1980) is solved.

Keywords: finite group; subgroup functor; τ-closed formation; formation σ-function; σ-local formation; critical σ-local formation; formation of classical type.

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Introduction

The problem of studying critical formations was posed by L. A. Shemetkov [1] at the 6th All-Union symposium on group theory. In the theory of local formations, a minimal local non-$\mathfrak{H}$-formation [1] or $\mathfrak{H}_l$-critical formation [2] is defined as a local formation $\mathfrak{F} \not\subseteq \mathfrak{H}$, all proper local subformations of which are contained in the class of groups $\mathfrak{H}$. The solution to L. A. Shemetkov's problem for local formations was obtained by A. N. Skiba in a series of works from 1980–1993 and culminated in the construction of the theory of critical local formations. The most general result was the description of minimal local non-$\mathfrak{H}$-formations for the case when $\mathfrak{H}$ is an arbitrary formation of classical type [3], i. e. a formation that has a local screen all of whose non-Abelian values are local. The theory of τ-closed σ-local formations was constructed by A. N. Skiba in the monograph [4]. Results on critical formations were widely used in studying the internal structure and classification of local and τ-closed local formations, as well as in studying their irreducible factorisations [4; 5].

The development of the theory of σ-local formations has led to the necessity of studying and classifying critical σ-local and functor-closed or, in other words, τ-closed σ-local formations, where τ is a subgroup functor in the sense of A. N. Skiba [4]. In this context, following [1; 2], a *minimal τ-closed σ-local non-$\mathfrak{H}$-formation* or *$\mathfrak{H}_\sigma^\tau$-critical formation* is defined as a τ-closed σ-local formation $\mathfrak{F} \not\subseteq \mathfrak{H}$, all proper τ-closed σ-local subformations of which are contained in the class of groups $\mathfrak{H}$. It should be noted that L. A. Shemetkov's problem of classifying critical formations in the class of σ-local formations was solved in the works [6; 7] for an arbitrary σ-local formation $\mathfrak{H}$ of classical type. Here, a formation $\mathfrak{H}$ is called [7] a *σ-local formation of classical type* if it has a σ-local definition all of whose non-Abelian values are σ-local.

The study of $\mathfrak{H}_\sigma^\tau$-critical formations was initiated by the authors in the work [8], where a criterion for minimal τ-closed σ-local non-$\mathfrak{H}$-formations was proved, and a description of such formations was given for a number of σ-local formations of classical type, in particular, for formations of all σ-nilpotent and all σ-soluble groups.

The main result of this paper is the following theorem, which provides a solution to the aforementioned problem of L. A. Shemetkov in the class of τ-closed σ-local formations.

Theorem 1. *Let $\mathfrak{H}$ be a σ-local formation of classical type and h be its canonical σ-local definition. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $G = P$ is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) \subseteq \{1, G\}$;

2) P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is either a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with monolith $Q = K^{h(\sigma_i)} \not\subseteq \Phi(K)$, where $\sigma_i \notin \sigma(Q)$, or a minimal non-$h(\sigma_i)$-group of one of the following types:

a) the quaternion group of order 8, if $2 \notin \sigma_i$;

b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$;

c) a cyclic q-group, $q \notin \sigma_i$.

Notations and definitions

The main definitions, notations of the theory of σ-properties of groups and general properties of σ-local formations are presented in the works [9–16].

Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, i. e. $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$, G be a group, $\mathfrak{F}$ be a class of groups. Then $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \varnothing\}$; $\sigma(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \sigma(G)$.

A group G is called [9]: *σ-primary*, if G is a σ_i-group for some i; *σ-nilpotent*, if $G = G_1 \times \ldots \times G_t$ for some σ-primary groups $G_1, \ldots, G_t$; *σ-soluble*, if $G = 1$ or $G \neq 1$ and every chief factor of G is σ-primary.

By $\mathfrak{S}_\sigma$, $\mathfrak{N}_\sigma$, $\mathfrak{G}_{\sigma_i}$ and (1) we denote the classes of all σ-soluble, σ-nilpotent, σ_i-groups and identity groups, respectively.

Let $\mathfrak{F}$ be a non-empty formation. Then for any group G the symbol $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of the group G, i. e. the smallest normal subgroup of G such that the quotient group belongs to the formation $\mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{H}$ are formations, then their *product* $\mathfrak{F}\mathfrak{H}$ is defined as follows: $\mathfrak{F}\mathfrak{H} = (G \mid G^{\mathfrak{H}} \in \mathfrak{F})$.

The symbol $O_{\sigma_i', \sigma_i}(G)$ denotes the $(\mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i})$-radical of the group G (the largest normal $(\mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i})$-subgroup of the group G), i. e. $O_{\sigma_i', \sigma_i}(G) = G_{\mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i}}$.

A function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called [10] a *formation σ-function*. For every formation σ-function f the class $LF_\sigma(f)$ is defined as follows:

$$LF_\sigma(f) = \left(G \mid G = 1 \text{ or } G \neq 1 \text{ and } G / O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)\right).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then it is said that the formation $\mathfrak{F}$ is *σ-local*, and f is a *σ-local definition* of $\mathfrak{F}$.

Suppose that to each group G a certain system of its subgroups $\tau(G)$ is assigned. Then τ is called a *subgroup functor* [4, p. 16], if $G \in \tau(G)$ and for any epimorphism $\varphi : A \to B$ and for any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $H^{\varphi^{-1}} \in \tau(A)$. A class of groups $\mathfrak{F}$ is called τ-closed, if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$.

A subgroup functor τ is called [4, p. 16]: *closed*, if for any two groups G and $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$; *trivial*, if for any group G we have $\tau(G) = \{G\}$; *identity*, if for any group G we have $\tau(G)$ is the set of all subgroups of the group G.

On the set of all subgroup functors, a partial order (≤) is introduced [4, p. 19], assuming that $\tau_1 \leq \tau_2$ holds if and only if for any group G the inclusion $\tau_1(G) \subseteq \tau_2(G)$ is valid. For any collection of subgroup functors $\{\tau_i \mid i \in I\}$ their intersection $\tau = \bigcap_{i \in I} \tau_i$ is defined: $\tau(G) = \bigcap_{i \in I} \tau_i(G)$ for any group G.

Let now τ be an arbitrary subgroup functor, $\overline{\tau}$ be the intersection of all such closed subgroup functors τ_i, for which $\tau \leq \tau_i$. The functor $\overline{\tau}$ is called the *closure of the functor* τ [4, p. 20].

For any set of groups $\mathfrak{X}$ the symbol $\mathrm{S}_\tau\mathfrak{X}$ [4, p. 23] denotes the set of all groups H, such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. It is clear that a class of groups $\mathfrak{F}$ is τ-closed, if $\mathrm{S}_\tau(\mathfrak{F}) = \mathfrak{F}$.

If f is a formation σ-function, then by $\mathrm{Supp}(f)$ we denote the *support* of f, i. e. the set of all σ_i such that $f(\sigma_i) \neq \varnothing$.

A formation σ-function f is called: *τ-valued*, if $f(\sigma_i)$ is a τ-closed formation for all $\sigma_i \in \mathrm{Supp}(f)$; *integrated*, if $f(\sigma_i) \subseteq LF_\sigma(f)$ for all i; *full*, if $f(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all i. If f is a full integrated formation σ-function and $\mathfrak{F} = LF_\sigma(f)$, then f is called the *canonical σ-local definition* of $\mathfrak{F}$. The *smallest σ-local definition* of $\mathfrak{F}$ is its σ-local definition f such that $f(\sigma_i) \subseteq h(\sigma_i)$ for any σ-local definition h of $\mathfrak{F}$ and all i.

Let $\mathfrak{H}$ be an arbitrary class of groups. A group G is called [4, p. 38] a *τ-minimal non-$\mathfrak{H}$-group* or a $\mathfrak{H}^\tau$*-critical group*, if $G \notin \mathfrak{H}$, but every proper τ-subgroup of the group G belongs to the class $\mathfrak{H}$.

For an arbitrary set of groups $\mathfrak{X}$ and any $\sigma_i \in \sigma$ the symbol $\mathfrak{X}(\sigma_i)$ [13, p. 962] denotes the class of groups defined as follows: $\mathfrak{X}(\sigma_i) = \left(G / O_{\sigma_i', \sigma_i}(G) \mid G \in \mathfrak{X}\right)$, if $\sigma_i \in \sigma(\mathfrak{X})$, $\mathfrak{X}(\sigma_i) = \varnothing$, if $\sigma_i \notin \sigma(\mathfrak{X})$.

The set of all τ-closed σ-local formations is denoted by l_σ^τ. Formations from l_σ^τ are called l_σ^τ*-formations.* For any set of groups $\mathfrak{M}$ the symbol $l_\sigma^\tau \operatorname{form} \mathfrak{M}$ denotes the τ-closed σ-local formation generated by the class of groups $\mathfrak{M}$, i. e. the intersection of all l_σ^τ-formations containing $\mathfrak{M}$. In the case when $\mathfrak{M} = \{G\}$ the formation $l_\sigma^\tau \operatorname{form} G$ is called a *one-generated* l_σ^τ*-formation.*

Let now $\{\mathfrak{F}_j \mid j \in J\}$ be some set of τ-closed σ-local formations. Then we set $\vee_\sigma^\tau(\mathfrak{F}_j \mid j \in J) = l_\sigma^\tau \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$. In particular, if $\mathfrak{H}$ and $\mathfrak{F}$ are τ-closed σ-local formations, then $\mathfrak{H} \vee_\sigma^\tau \mathfrak{F} = l_\sigma^\tau \operatorname{form}(\mathfrak{H} \cup \mathfrak{F})$.

Let $\mathfrak{F}$ be a τ-closed σ-local formation. The formation $\mathfrak{F}$ is called an *irreducible* τ-*closed* σ-*local formation* or an l_σ^τ-*irreducible formation*, if $\mathfrak{F} \neq l_\sigma^\tau \operatorname{form}(\cup_{i \in I} \mathfrak{X}_i) = \vee_\sigma^\tau(\mathfrak{X}_i \mid i \in I)$, where $\{\mathfrak{X}_i \mid i \in I\}$ is the set of all proper τ-closed σ-local subformations of $\mathfrak{F}$. If there exist proper τ-closed σ-local subformations $\mathfrak{X}$ and $\mathfrak{H}$ of $\mathfrak{F}$, such that $\mathfrak{F} = \mathfrak{X} \vee_\sigma^\tau \mathfrak{H}$, then the formation $\mathfrak{F}$ is called a *reducible* τ-*closed* σ-*local formation* or l_σ^τ-*reducible formation*.

Auxiliary results

For the proof of the main result of the paper, we will need the following known facts from formation theory.

Lemma 1 [13, lemma 2.1]. *Let f and h be formation* σ-*functions and let* $\Pi = \operatorname{Supp}(f)$. *Suppose that* $\mathfrak{F} = LF_\sigma(f) = LF_\sigma(h)$. *Then:*

(1) $\Pi = \sigma(\mathfrak{F})$;

(2) $\mathfrak{F} = \left(\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{G}_\Pi$. *Consequently,* $\mathfrak{F}$ *is a saturated formation;*

(3) if every group in $\mathfrak{F}$ *is* σ-*soluble, then* $\mathfrak{F} = \left(\bigcap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{G}_\Pi$;

(4) if $\sigma_i \in \Pi$, *then* $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$;

(5) $\mathfrak{F} = LF_\sigma(F)$, *where F is the unique formation* σ-*function such that* $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ *for all* $\sigma_i \in \Pi$ *and* $F(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$. *Moreover,* $F(\sigma_i) = \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})$ *for all i.*

Lemma 2 [17, lemma 1.1 (a)]. *Let* $\mathfrak{F}$ *be a Fitting class and G be a finite group. If* $N \lhd\lhd G$, *then* $N_\mathfrak{F} = N \cap G_\mathfrak{F}$.

Lemma 3 [18, lemma 1.2]. *Let* $\mathfrak{F}$ *be a non-empty formation,* $K \trianglelefteq G$. *Then the following statements hold:*

(1) $(G/K)^\mathfrak{F} = G^\mathfrak{F} K/K$;

(2) if $G = HK$, *then* $H^\mathfrak{F} K = G^\mathfrak{F} K$;

(3) if $G = HK$ *and* $K \subseteq G^\mathfrak{F}$, *then* $H^\mathfrak{F} K = G^\mathfrak{F}$.

Lemma 4 [16, lemma 22]. *Let* $\mathfrak{M} = LF_\sigma(m)$, *where m is a formation* σ-*function such that* $m(\sigma_i)$ *is a* τ-*closed formation for all* $\sigma_i \in \operatorname{Supp}(m)$. *Then the formation* $\mathfrak{M}$ *is* τ-*closed.*

Recall that $\tau \operatorname{form} \mathfrak{X}$ denotes the τ-closed formation generated by the class of groups $\mathfrak{X}$, i. e. the intersection of all τ-closed formations containing $\mathfrak{X}$.

Lemma 5 [15, theorem 1.1]. *Let* $\mathfrak{X}$ *be some non-empty set of groups,* $\mathfrak{F} = l_\sigma^\tau \operatorname{form} \mathfrak{X} = LF_\sigma(f)$, *where f is the smallest* τ-*valued* σ-*local definition of* $\mathfrak{F}$. *Then the following statements hold:*

(1) $\sigma(\mathfrak{X}) = \sigma(\mathfrak{F})$;

(2) $f(\sigma_i) = \tau \operatorname{form}(\mathfrak{X}(\sigma_i)) = \tau \operatorname{form}(\mathfrak{F}(\sigma_i))$ *for all i;*

(3) if h is an arbitrary τ-*valued* σ-*local definition of* $\mathfrak{F}$, *then for all* $\sigma_i \in \sigma(\mathfrak{X})$ *we have* $f(\sigma_i) = \tau \operatorname{form}(A \mid A \in \mathfrak{F} \cap h(\sigma_i),\ O_{\sigma_i}(A) = 1)$.

Following [18], we define a partial order (≤) on the set of all formation σ-functions as follows: if f_1 and f_2 are formation σ-functions, then $f_1 \leq f_2$, if $f_1(\sigma_i) \subseteq f_2(\sigma_i)$ for any $\sigma_i \in \sigma$.

The following lemma is a special case of corollary 3.1 from paper [15].

Lemma 6 [15, corollary 3.1]. *Let* $\mathfrak{F}_j = LF_\sigma(f_j)$, *where* f_j *is the smallest* τ-*valued* σ-*local definition of the formation* $\mathfrak{F}_j$, $j = 1, 2$. *Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $f_1 \leq f_2$.

Lemma 7 [14, lemma 11]. *If* $\mathfrak{F} = LF_\sigma(f)$ *and* $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{F}$ *for some* $\sigma_i \in \sigma(G)$, *then* $G \in \mathfrak{F}$.

A τ-closed formation $\mathfrak{F}$ is called [4] an *irreducible* τ-*closed formation* or a τ-*irreducible formation*, if $\mathfrak{F} \neq \tau \operatorname{form}(\cup_{i \in I} \mathfrak{X}_i)$, where $\{\mathfrak{X}_i \mid i \in I\}$ is the set of all proper τ-closed subformations of $\mathfrak{F}$.

Lemma 8 [4, lemma 2.1.5]. *Let A be a monolithic group with monolith* $R \not\subseteq \Phi(A)$. *Then the formation* $\mathfrak{F} = \tau \operatorname{form} A$ *is* τ-*irreducible and it has a unique maximal* τ-*closed subformation* $\mathfrak{M} = \tau \operatorname{form}(\{A/R\} \cup \mathfrak{X})$, *where* $\mathfrak{X}$ *is the set of all proper* τ-*subgroups of A.*

Lemma 9 [7, lemma 2.1]. *Let* $\varnothing \neq \Pi \subseteq \sigma$. *Then the class* $\mathfrak{G}_\Pi$ *of all* Π-*groups and the class* $\mathfrak{N}_\Pi$ *of all* σ-*nilpotent* Π-*groups are* σ-*local formations and the following statements hold:*

(1) $\mathfrak{G}_\Pi = LF_\sigma(g)$, *where g is the canonical* σ-*local definition of the formation* $\mathfrak{G}_\Pi$. *Here,* $g(\sigma_i) = \mathfrak{G}_\Pi$ *for all* $\sigma_i \in \Pi$ *and* $g(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$;

(2) $\mathfrak{N}_\Pi = LF_\sigma(n) = LF_\sigma(k)$*, where n and k, respectively, are the smallest and canonical* σ*-local definitions of the formation* $\mathfrak{N}_\Pi$*. Here,* $n(\sigma_i) = (1)$ *for all* $\sigma_i \in \Pi$ *and* $n(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$*,* $k(\sigma_i) = \mathfrak{G}_{\sigma_i}$ *for all* $\sigma_i \in \Pi$ *and* $k(\sigma_i) = \varnothing$ *for all* $\sigma_i \in \Pi'$*.*

Lemma 10 [13, theorem 1.14]. *Let* $\mathfrak{M} = LF_\sigma(m)$*, where m is integrated,* $\Pi = \sigma(\mathfrak{M})$*, and let* $\mathfrak{H}$ *be a non-empty formation. Suppose that either (i)* $\mathfrak{H} = LF_\sigma(h)$*, where h is integrated, or (ii)* $\mathfrak{G}_{\sigma_i}\mathfrak{H} \subseteq \mathfrak{H}$ *for all* $\sigma_i \in \sigma \setminus \Pi$*. Then* $\mathfrak{M}\mathfrak{H} = LF_\sigma(f)$*, where*

$$f(\sigma_i) = \begin{cases} m(\sigma_i)\mathfrak{H}, \ \text{if } \sigma_i \in \Pi, \\ h(\sigma_i), \ \text{if } \sigma_i \in \sigma \setminus \Pi \text{ in case } (i), \\ \mathfrak{H}, \ \text{if } \sigma_i \in \sigma \setminus \Pi \text{ in case } (ii). \end{cases}$$

Lemma 11 [8, theorem 3.1]. *Let* $\mathfrak{H} = LF_\sigma(h)$*,* $\mathfrak{F} = LF_\sigma(f)$*, where h is the canonical* σ*-local definition of* $\mathfrak{H}$*, f is the smallest* τ*-valued* σ*-local definition of* $\mathfrak{F}$*. Then* $\mathfrak{F}$ *is a minimal* τ*-closed* σ*-local non-*$\mathfrak{H}$*-formation if and only if* $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$*, where G is a monolithic* $\bar{\tau}$*-minimal non-*$\mathfrak{H}$*-group with monolith* $P = G^{\mathfrak{H}}$*, such that for all* $\sigma_i \in \sigma(P)$ *the formation* $f(\sigma_i)$ *is a minimal* τ*-closed non-*$h(\sigma_i)$*-formation.*

Lemma 12 [5, corollary 7.14]. *The set of n-multiply local formations is a subsemigroup in the semigroup* $G\mathfrak{G}$ *of all formations.*

Lemma 13 [5, lemma 18.8]. *If a group G has only one minimal normal subgroup and* $O_p(G) = 1$ *(p is some prime), then there exists a faithful irreducible* F_pG*-module, where* F_p *is the field of p elements.*

Lemma 14 [4, corollary 1.2.23]. *Let* $\mathfrak{X}$ *be an arbitrary set of groups,* $\mathfrak{M} = \mathrm{S}_{\bar{\tau}}(\mathfrak{X})$*. Then* $\tau \operatorname{form} \mathfrak{X} = \operatorname{form} \mathfrak{M}$.

The following lemma is a consequence of lemmas 1 and 6.

Lemma 15. *Let* $\mathfrak{F}_i = LF_\sigma(F_i)$*, where* F_i *is the canonical* σ*-local definition of* $\mathfrak{F}_i$*,* $i = 1, 2$*. Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $F_1 \leq F_2$*.*

Lemma 16 [5, corollary 19.14]. *A group is nilpotent if and only if all subformations of the formation it generates are hereditary.*

Lemma 17 [4, lemma 2.2.4]. *A formation* $\mathfrak{F}$ *is a minimal* τ*-closed non-Abelian formation if and only if* $\mathfrak{F} = \tau \operatorname{form} A$*, where A is a* τ*-minimal non-Abelian group and one of the following conditions holds:*

1) A is a monolithic group with a non-Frattini monolith $R = A'$*;*

2) A is the quaternion group of order 8;

3) A is a nilpotent group of class 2 and its exponent is 4;

4) A is an extraspecial p-group of order p^3 *and its exponent is* $p > 2$*;*

5) A is a nilpotent group of class 2 having a prime odd exponent.

Recall that a group $G \in \mathrm{R}_0\mathfrak{X}$ [17, p. 264] if and only if G has normal subgroups $N_1, \ldots, N_t$ $(t \geq 2)$ such that $G / N_i \in \mathfrak{X}$ and $\cap_{i=1}^t N_i = 1$. The set of all homomorphic images of all groups from $\mathfrak{X}$ is denoted by $\mathrm{H}\mathfrak{X}$.

Lemma 18 [4, lemma 1.2.22]. *For any set of groups* $\mathfrak{X}$ *the following equality holds:*

$$\tau \operatorname{form} \mathfrak{X} = \mathrm{HR}_0\mathrm{S}_{\bar{\tau}}(\mathfrak{X}).$$

Lemma 19 [5, lemma 8.12]. *Let* $A \in s \operatorname{form} G$*, where G is a finite group. Then the following statements hold:*

(1) the exponent of the group A does not exceed the exponent of the group G;

(2) every chief factor of the group A is isomorphic to some chief factor of the group G;

(3) every composition factor of the group A is isomorphic to some composition factor of the group G;

(4) the class of any nilpotent factor of the group A does not exceed the maximum of the classes of the nilpotent factors of the group G.

Lemma 20 [12, lemma 2.4]. *If* $\mathfrak{F} = LF_\sigma(f)$*, then* $\mathfrak{F} = LF_\sigma(t)$*, where* $t(\sigma_i) = f(\sigma_i) \cap \mathfrak{F}$ *for all* $\sigma_i \in \sigma$*.*

Lemma 21 [8, theorem 4.1]. *Let* $\mathfrak{F}$ *be a* τ*-closed* σ*-local formation,* $\mathfrak{F} \not\subseteq \mathfrak{G}_\Pi$*, where* $\varnothing \neq \Pi \subseteq \sigma$*. Then* $\mathfrak{F}$ *is a minimal* τ*-closed* σ*-local non-*$\mathfrak{G}_\Pi$*-formation if and only if* $\mathfrak{F} = \mathfrak{G}_{\sigma_i} = l_\sigma^\tau \operatorname{form} G$*, where G is a simple* σ_i*-group such that* $\sigma_i \notin \Pi$ *and* $\tau(G) \subseteq \{1, G\}$*.*

Main result

Before we prove theorem 1, we establish the validity of the following lemmas.

Lemma 22. *Let* $\mathfrak{F} = LF_\sigma(f)$*, where f is the canonical* σ*-local definition of* $\mathfrak{F}$ *and let* $\mathfrak{H}$ *be a non-empty formation such that* $\sigma(\mathfrak{H}) \subseteq \sigma(\mathfrak{F})$*. Then* $\mathfrak{F}\mathfrak{H} = LF_\sigma(h)$*, where h is the canonical* σ*-local definition of* $\mathfrak{F}\mathfrak{H}$*, and the following statements hold:*

(1) $h(\sigma_i) = f(\sigma_i)\mathfrak{H}$*, if* $\sigma_i \in \sigma(\mathfrak{F})$*;*

(2) $h(\sigma_i) = \varnothing$*, if* $\sigma_i \notin \sigma(\mathfrak{F})$*.*

P r o o f. First, note that $h(\sigma_i) \subseteq \mathfrak{F}\mathfrak{H}$ for any $\sigma_i \in \sigma(\mathfrak{F})$. Let $\mathfrak{M} = LF_\sigma(h)$. By lemma 1 (1) we have $\sigma(\mathfrak{M}) = \sigma(\mathfrak{F})$.

Assume that $\mathfrak{M} \not\subseteq \mathfrak{F}\mathfrak{H}$ and let G be a group of minimal order from $\mathfrak{M} \setminus \mathfrak{F}\mathfrak{H}$. Then G is a monolithic group with monolith $P = G^{\mathfrak{F}\mathfrak{H}}$.

First, consider the case when P is a σ-primary group. Then P is a σ_i-group for some $\sigma_i \in \sigma(\mathfrak{F})$ and $O_{\sigma_{i'},\sigma_i}(G) = O_{\sigma_i}(G)$. Since $G \in \mathfrak{M}$, we have

$$G / O_{\sigma_i}(G) = G / O_{\sigma_{i'},\sigma_i}(G) \in h(\sigma_i) = f(\sigma_i)\mathfrak{H} = \mathfrak{G}_{\sigma_i} f(\sigma_i)\mathfrak{H}.$$

Therefore, $G \in f(\sigma_i)\mathfrak{H} \subseteq \mathfrak{F}\mathfrak{H}$, a contradiction. Hence, the group P is non-σ-primary. Then $O_{\sigma_{i'},\sigma_i}(G) = 1$ for any $\sigma_i \in \sigma(P)$. Consequently, $G \simeq G / O_{\sigma_{i'},\sigma_i}(G) \in f(\sigma_i) \subseteq \mathfrak{F}\mathfrak{H}$ and we again obtain a contradiction with the choice of the group G. Hence, this case is impossible. Therefore, $\mathfrak{M} \subseteq \mathfrak{F}\mathfrak{H}$.

Now suppose that $\mathfrak{F}\mathfrak{H} \not\subseteq \mathfrak{M}$ and let A be a group of minimal order from $\mathfrak{F}\mathfrak{H} \setminus \mathfrak{M}$. Then A is a monolithic group with monolith $R = A^{\mathfrak{M}}$. Since $A \in \mathfrak{F}\mathfrak{H}$, we have $A^{\mathfrak{H}} \in \mathfrak{F}$. Also note that since $\mathfrak{H} \subseteq \mathfrak{F}\mathfrak{H}$, we have $A^{\mathfrak{H}} \neq 1$. Therefore, $R \leq A^{\mathfrak{H}}$. Since $A^{\mathfrak{H}} \in \mathfrak{F}$, we have $A^{\mathfrak{H}} / O_{\sigma_{i'},\sigma_i}(A^{\mathfrak{H}}) \in f(\sigma_i)$. Since $O_{\sigma_{i'},\sigma_i}(A^{\mathfrak{H}}) = O_{\sigma_{i'},\sigma_i}(A) \cap A^{\mathfrak{H}}$ by lemma 2, we have

$$O_{\sigma_{i'},\sigma_i}(A) A^{\mathfrak{H}} / O_{\sigma_{i'},\sigma_i}(A) \simeq A^{\mathfrak{H}} / O_{\sigma_{i'},\sigma_i}(A) \cap A^{\mathfrak{H}} = A^{\mathfrak{H}} / O_{\sigma_{i'},\sigma_i}(A^{\mathfrak{H}}) \in f(\sigma_i).$$

But by lemma 3 (1) we have the equality $O_{\sigma_{i'},\sigma_i}(A) A^{\mathfrak{H}} / O_{\sigma_{i'},\sigma_i}(A) = \left(A / O_{\sigma_{i'},\sigma_i}(A)\right)^{\mathfrak{H}}$. Therefore, $A / O_{\sigma_{i'},\sigma_i}(A) \in f(\sigma_i)\mathfrak{H} = h(\sigma_i)$ for any $\sigma_i \in \sigma(R)$. Moreover, since $A / R \in \mathfrak{M}$ and $O_{\sigma_{i'},\sigma_i}(A / R) = O_{\sigma_{i'},\sigma_i}(A) / R$ for any $\sigma_i \in \sigma(A) \setminus \sigma(R)$, we have

$$A / O_{\sigma_{i'},\sigma_i}(A) \simeq (A / R) / O_{\sigma_{i'},\sigma_i}(A) / R = (A / R) / O_{\sigma_{i'},\sigma_i}(A / R) \in h(\sigma_i)$$

for any $\sigma_i \in \sigma(A) \setminus \sigma(R)$. Thus, $A / O_{\sigma_{i'},\sigma_i}(A) \in h(\sigma_i)$ for any $\sigma_i \in \sigma(A)$. Consequently, $A \in \mathfrak{M}$. The obtained contradiction shows that our assumption is false. Hence, $\mathfrak{F}\mathfrak{H} \subseteq \mathfrak{M}$ and therefore $\mathfrak{F}\mathfrak{H} = LF_\sigma(h)$. Since f is the canonical σ-local definition of the formation $\mathfrak{F}$, for any σ_i the equality

$$\mathfrak{G}_{\sigma_i} h(\sigma_i) = \mathfrak{G}_{\sigma_i}\left(f(\sigma_i)\mathfrak{H}\right) = \left(\mathfrak{G}_{\sigma_i} f(\sigma_i)\right)\mathfrak{H} = f(\sigma_i)\mathfrak{H} = h(\sigma_i)$$

holds. Since also $h(\sigma_i) \subseteq LF_\sigma(h) = \mathfrak{F}\mathfrak{H}$ for any σ_i, it follows that h is the canonical σ-local definition of the formation $\mathfrak{F}\mathfrak{H}$. The lemma is proved.

Lemma 23. *Let $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and H is a monolithic group with monolith Q, such that $\sigma_i \notin \sigma(Q)$ and $\Phi(H) = 1$. And let $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of H. Then $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$ is l_σ^τ-irreducible and its maximal τ-closed σ-local subformation $\mathfrak{H}$ has an integrated τ-valued σ-local definition h such that $h(\sigma_i) = \tau\operatorname{form}(\mathfrak{X} \cup \{H / Q\})$; $h(\sigma_j) = \tau\operatorname{form}\left(G / O_{\sigma_{j'},\sigma_j}(G)\right)$ for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$ and $h(\sigma_j) = \varnothing$ for all $\sigma_j \notin \sigma(G)$.*

P r o o f. Let $\mathfrak{H} = LF_\sigma(h)$, where h is the formation σ-function from the condition of the lemma. Then by lemma 4 the formation $\mathfrak{H}$ is τ-closed. Let us show that $G / P \in \mathfrak{H}$. Let $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$. Then $P \subseteq O_{\sigma_{j'}}(G)$. Hence, $O_{\sigma_{j'},\sigma_j}(G / P) = O_{\sigma_{j'},\sigma_j}(G) / P$. Therefore,

$$G / O_{\sigma_{j'},\sigma_j}(G) \simeq (G / P) / \left(O_{\sigma_{j'},\sigma_j}(G) / P\right) = (G / P) / O_{\sigma_{j'},\sigma_j}(G / P).$$

But by the construction of h we have $h(\sigma_j) = \tau\operatorname{form}\left(G / O_{\sigma_{j'},\sigma_j}(G)\right)$. Consequently,

$$(G / P) / O_{\sigma_{j'},\sigma_j}(G / P) \in h(\sigma_j).$$

Let under the canonical isomorphism $H \simeq G / P$ the subgroup Q in the group G / P correspond to the subgroup Q_1 / P. Then, by the condition of the lemma, $\sigma_i \in \sigma(Q_1 / P)$. Hence, $Q_1 / P \subseteq O_{\sigma_{i'}}(G / P)$. Consequently, $Q_1 / P \subseteq O_{\sigma_{i'}}(G / P) \subseteq O_{\sigma_{i'},\sigma_i}(G / P)$. But

$$(G / P) / (Q_1 / P) \simeq H / Q \in \tau\operatorname{form}(\mathfrak{X} \cup \{H / Q\}) = h(\sigma_i).$$

Hence, $(G / P) / O_{\sigma_{i'},\sigma_i}(G / P) \in h(\sigma_i)$. Thus, for any $\sigma_j \in \sigma(G / P)$ we have $(G / P) / O_{\sigma_{j'},\sigma_j}(G / P) \in h(\sigma_j)$. Therefore, $G / P \in LF_\sigma(h) = \mathfrak{H}$.

Let us show that h is an integrated σ-local definition of the formation $\mathfrak{H}$. By the construction of the formation σ-function h we have

$$h(\sigma_i) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{H/Q\}\right), \tag{1}$$

where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of the group H. It is clear that in equality (1) one can take as $\mathfrak{X}$ the set of all proper $\overline{\tau}$-subgroups from G/P. But, as was established earlier, $G/P \in \mathfrak{H}$ and the formation $\mathfrak{H}$ is τ-closed. Therefore, $\mathfrak{X} \subseteq \mathfrak{H}$. Consequently, $h(\sigma_i) \subseteq \mathfrak{H}$. Now let $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$. Then $O_{\sigma_j', \sigma_j}(G)/P = O_{\sigma_j', \sigma_j}(G/P)$. Hence, since $G/O_{\sigma_j', \sigma_j}(G) \simeq (G/P)/O_{\sigma_j', \sigma_j}(G/P)$ and $G/P \in \mathfrak{H}$, we have $G/O_{\sigma_j', \sigma_j}(G) \in \mathfrak{H}$. Therefore,

$$h(\sigma_j) = \tau\,\mathrm{form}\left(G/O_{\sigma_j', \sigma_j}(G)\right) \subseteq \mathfrak{H}.$$

Thus, the σ-function h is integrated.

Let $\mathfrak{M}$ be an arbitrary proper τ-closed σ-local subformation of $\mathfrak{F}$, m be its smallest τ-valued σ-local definition. Let us show that $\mathfrak{M} \subseteq \mathfrak{H}$.

Denote by f the smallest τ-valued σ-local definition of $\mathfrak{F}$. Then by lemma 5 for every $\sigma_j \in \sigma(G)$ we have $f(\sigma_j) = \tau\,\mathrm{form}\left(G/O_{\sigma_j', \sigma_j}(G)\right)$ and $f(\sigma_j) = \varnothing$ for all $\sigma_j \notin \sigma(G)$. By lemma 6 we have $m \le f$. Let $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$. Then by the construction of h we have $h(\sigma_j) = f(\sigma_j)$. Hence, for all such σ_j the inclusion $m(\sigma_j) \subseteq h(\sigma_j)$ holds. Now consider the value of f on σ_i. If $m(\sigma_i) = f(\sigma_i)$, then $G \in \mathfrak{M}$. Indeed, since $O_{\sigma_i', \sigma_i}(G) = O_{\sigma_i}(G)$ and $O_{\sigma_i}(H) = 1$, then

$$O_{\sigma_i}(G) = O_{\sigma_i}(G) \cap G = O_{\sigma_i}(G) \cap PH = P\left(O_{\sigma_i}(G) \cap H\right) = P.$$

Therefore, $G/O_{\sigma_i}(G) = G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) = m(\sigma_i)$. Applying now lemma 7, we conclude that $G \in \mathfrak{M}$. Hence, $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}\,G \subseteq \mathfrak{M}$. The latter contradicts the definition of $\mathfrak{M}$. Thus, $m(\sigma_i) \subset f(\sigma_i)$. But by lemma 8 $\tau\,\mathrm{form}\left(\mathfrak{X} \cup \{H/Q\}\right) = h(\sigma_i)$ is the unique maximal τ-subformation of the τ-irreducible formation

$$f(\sigma_i) = \tau\,\mathrm{form}\left(G/O_{\sigma_i', \sigma_i}(G)\right) = \tau\,\mathrm{form}(G/P) = \tau\,\mathrm{form}\,H.$$

Consequently, $m(\sigma_i) \subseteq \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{H/Q\}\right) = h(\sigma_i)$. Thus, for all σ_j we have $m(\sigma_j) \subseteq h(\sigma_j)$. Hence, $\mathfrak{M} \subseteq \mathfrak{H}$.

Let us now show that $\mathfrak{H} \subseteq \mathfrak{F}$. Indeed, by lemma 5 we have

$$f(\sigma_i) = \tau\,\mathrm{form}(G/P) = \tau\,\mathrm{form}\,H.$$

Hence, $\mathfrak{X} \cup \{H/Q\} \subseteq f(\sigma_i)$. Therefore, $h(\sigma_i) \subseteq f(\sigma_i)$. On the other hand, by the construction of the formation σ-function h for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$ we have $h(\sigma_j) \subseteq f(\sigma_j)$. So, for all σ_j the inclusion $h(\sigma_j) \subseteq f(\sigma_j)$ holds. Hence, $\mathfrak{H} \subseteq \mathfrak{F}$. Assume that $\mathfrak{H} = \mathfrak{F}$. Then $t = f$, where t is the smallest τ-valued σ-local definition of $\mathfrak{H}$. Moreover, $t \le h$. But $h(\sigma_i) \subset f(\sigma_i)$. All the more $t(\sigma_i) \subset f(\sigma_i)$. The obtained contradiction shows that $\mathfrak{H} \ne \mathfrak{F}$. The lemma is proved.

Recall that for any $\Pi \subseteq \sigma$ the symbol Π' denotes the set $\sigma \setminus \Pi$.

Lemma 24. *Let $\mathfrak{M}$ be a non-empty formation, $\varnothing \ne \Pi \subseteq \sigma$. Then the canonical σ-local definition h of $\mathfrak{H} = \mathfrak{G}_{\Pi'}\mathfrak{N}_\Pi\mathfrak{M}$ is such that $h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}$ for all $\sigma_i \in \Pi$ and $h(\sigma_i) = \mathfrak{H}$ for all $\sigma_i \in \Pi'$.*

P r o o f. Let $\mathfrak{F} = \mathfrak{G}_{\Pi'}\mathfrak{N}_\Pi$. By lemma 9 (1) the formation $\mathfrak{G}_{\Pi'}$ has a σ-local definition t such that $t(\sigma_i) = \mathfrak{G}_{\Pi'}$ for all $\sigma_i \in \Pi'$ and $t(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi$. Moreover, by lemma 9 (2) the formation $\mathfrak{N}_\Pi$ has a σ-local definition k such that $k(\sigma_i) = \mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \Pi$ and $k(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Hence, by lemma 10 (i) the formation $\mathfrak{F}$ has a σ-local definition f such that $f(\sigma_i) = \mathfrak{G}_{\Pi'}\mathfrak{N}_\Pi$ for all $\sigma_i \in \Pi'$ and $f(\sigma_i) = \mathfrak{G}_{\sigma_i}$ for all $\sigma_i \in \sigma$. Applying now lemma 22, we see that $\mathfrak{H}$ has a σ-local definition h such that $h(\sigma_i) = f(\sigma_i)\mathfrak{M} = \mathfrak{G}_{\Pi'}\mathfrak{N}_\Pi\mathfrak{M}$, if $\sigma_i \in \Pi'$, and $h(\sigma_i) = f(\sigma_i)\mathfrak{M} = \mathfrak{G}_{\sigma_i}\mathfrak{M}$, if $\sigma_i \in \Pi$. Since, obviously, $f(\sigma_i) \subseteq \mathfrak{H}$ and $h(\sigma_i) = \mathfrak{G}_{\sigma_i}h(\sigma_i)$ for all i, it follows that h is the canonical σ-local definition of $\mathfrak{H}$. The lemma is proved.

Lemma 25. *Let A be a monolithic group with a non-σ-primary monolith P, $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of the group A. Then the formation $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}\,A$ is l_σ^τ-irreducible, and its maximal τ-closed σ-local subformation $\mathfrak{H}$ has an integrated σ-local definition h such that $h(\sigma_i) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A/P\}\right)$ for all $\sigma_i \in \sigma(P)$, $h(\sigma_i) = \tau\,\mathrm{form}\left(A/O_{\sigma_i', \sigma_i}(A)\right)$ for all $\sigma_i \in \sigma(A) \setminus \sigma(P)$ and $h(\sigma_i) = \varnothing$ for all $\sigma_i \notin \sigma(A)$.*

P r o o f. Let h be the formation σ-function from the condition of the lemma, and $\mathfrak{H} = LF_\sigma(h)$. Then by lemma 4 the formation $\mathfrak{H}$ is τ-closed. Let us show that $\left(\mathfrak{X} \cup \{A/P\}\right) \subseteq \mathfrak{H}$. Let $G \in \left(\mathfrak{X} \cup \{A/P\}\right)$, $\sigma_i \in \sigma(G)$. If $\sigma_i \in \sigma(P)$, then $h(\sigma_i) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A/P\}\right)$ and therefore $G/O_{\sigma_i', \sigma_i}(G) \in h(\sigma_i)$. Let $\sigma_i \in \sigma(G) \setminus \sigma(P)$. Then since $G \in \mathfrak{F}$, we have $G / O_{\sigma_{i'}, \sigma_i}(G) \in f(\sigma_i)$, where f is the smallest τ-valued σ-local definition of $\mathfrak{F}$. By lemma 5 we have $f(\sigma_i) = \tau\,\mathrm{form}\left(A / O_{\sigma_{i'}, \sigma_i}(A)\right) = h(\sigma_i)$. Hence, $G / O_{\sigma_{i'}, \sigma_i}(G) \in h(\sigma_i)$. Thus, for all $\sigma_i \in \sigma(G)$ we have $G / O_{\sigma_{i'}, \sigma_i}(G) \in h(\sigma_i)$. This means that $G \in LF_\sigma(h) = \mathfrak{H}$. Consequently, $\mathfrak{X} \cup \{A / P\} \subseteq \mathfrak{H}$. From this it follows that for all $\sigma_i \in \sigma(P)$ we have $h(\sigma_i) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A / P\}\right) \subseteq \mathfrak{H}$. Let $\sigma_i \in \sigma(A) \setminus \sigma(P)$. Then $P \subseteq O_{\sigma_{i'}, \sigma_i}(A)$ and therefore $A / O_{\sigma_{i'}, \sigma_i}(A)$ is a homomorphic image of A / P. Hence, $A / O_{\sigma_{i'}, \sigma_i}(A) \subseteq \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A / P\}\right) \subseteq \mathfrak{H}$. The latter implies $h(\sigma_i) \subseteq \mathfrak{H}$. Therefore, the σ-local definition h is integrated.

Now let $\mathfrak{M}$ be an arbitrary proper τ-closed σ-local subformation of $\mathfrak{F}$, m be its smallest τ-valued σ-local definition. Let us show that $\mathfrak{M} \subseteq \mathfrak{H}$. By lemma 6 we have $m \leq f$. By lemma 5 for all $\sigma_i \in \sigma(A) \setminus \sigma(P)$ we have $h(\sigma_i) = f(\sigma_i)$. Hence, for all such σ_i the inclusion $m(\sigma_i) \subseteq h(\sigma_i)$ holds. Let $\sigma_i \in \sigma(P)$. Then $O_{\sigma_{i'}, \sigma_i}(A) = 1$ and by lemma 5 we obtain

$$f(\sigma_i) = \tau\,\mathrm{form}\left(A / O_{\sigma_{i'}, \sigma_i}(A)\right) = \tau\,\mathrm{form}(A).$$

If $f(\sigma_i) = m(\sigma_i)$, then $A \in m(\sigma_i)$. But m is an integrated σ-local definition of $\mathfrak{M}$. Hence, $A \in \mathfrak{M}$ and therefore $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}(A) \subseteq \mathfrak{M} \subset \mathfrak{F}$. Hence, $\mathfrak{F} = \mathfrak{M}$. The latter contradicts the definition of $\mathfrak{M}$. Thus, $m(\sigma_i) \subset f(\sigma_i)$. By lemma 8 the formation $f(\sigma_i) = \tau\,\mathrm{form}(A)$ is τ-irreducible and every its proper τ-subformation is contained in $\tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A / P\}\right) = h(\sigma_i)$. Hence, $m(\sigma_i) \subseteq h(\sigma_i)$. Thus, $m \leq h$ and therefore $\mathfrak{M} \subseteq \mathfrak{H}$.

By the construction of the σ-function h we have $h(\sigma_i) \subseteq f(\sigma_i)$ for all σ_i. Consequently, $\mathfrak{H} \subseteq \mathfrak{F}$. If $\sigma_i \in \sigma(P)$, then $h(\sigma_i) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{A / P\}\right) \subset f(\sigma_i)$. All the more $t(\sigma_i) \subset f(\sigma_i)$, where t is the smallest τ-valued σ-local definition of $\mathfrak{H}$. The latter means that $\mathfrak{H} \subset \mathfrak{F}$. Thus, the formation $\mathfrak{F}$ is l_σ^τ-irreducible and $\mathfrak{H}$ is its unique maximal τ-closed σ-local subformation. The lemma is proved.

Lemma 26. *Let $\mathfrak{M}$ be a saturated formation, $\mathfrak{H} = \mathfrak{G}_{\sigma_{i'}} \mathfrak{G}_{\sigma_i} \mathfrak{M}$. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}\,G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, such that $\sigma_i \in \sigma(P)$, and either P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i} \mathfrak{M}\right)$-group with $P = G^{\mathfrak{G}_{\sigma_i} \mathfrak{M}}$, or $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and H is a monolithic $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i} \mathfrak{M}\right)$-group with monolith $Q = H^{\mathfrak{G}_{\sigma_i} \mathfrak{M}}$, such that $Q \not\subseteq \Phi(H)$ and $\sigma_i \notin \sigma(Q)$.*

P r o o f. Let f be the smallest τ-valued σ-local definition of the formation $\mathfrak{F}$, h be the canonical σ-local definition of the formation $\mathfrak{H}$.

Necessity. By lemma 11 $\mathfrak{F}$ is a τ-closed σ-local formation generated by a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group G with monolith $P = G^{\mathfrak{H}}$, such that the formation $f(\sigma_j)$ is a minimal τ-closed non-$h(\sigma_j)$-formation for all $\sigma_j \in \sigma(P)$. Since $\mathfrak{H} = \mathfrak{G}_{\sigma_{i'}} \mathfrak{H}$, we have $\sigma_i \in \sigma(P)$.

Let P be a non-σ-primary group. Then by lemma 25 the formation $\mathfrak{F}$ is l_σ^τ-irreducible and its maximal τ-closed σ-local subformation $\mathfrak{M}_1$ has an integrated τ-valued σ-local definition m such that $m(\sigma_j) = \tau\,\mathrm{form}\left(\mathfrak{X} \cup \{G / P\}\right)$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of the group G for all $\sigma_j \in \sigma(P)$. But by condition $\mathfrak{M}_1 \subseteq \mathfrak{H}$. Hence, $m \leq h$. By lemma 24 we have $h(\sigma_i) = \mathfrak{G}_{\sigma_i} \mathfrak{M}$. Thus, $\left(\mathfrak{X} \cup \{G / P\}\right) \subseteq \mathfrak{G}_{\sigma_i} \mathfrak{M}$. It is clear that G does not belong to $\mathfrak{G}_{\sigma_i} \mathfrak{M}$. Hence, G is a $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i} \mathfrak{M}\right)$-group and $G / P \in \mathfrak{G}_{\sigma_i} \mathfrak{M}$, i. e. $P = G^{\mathfrak{G}_{\sigma_i} \mathfrak{M}}$.

Let P be a σ-primary group. Then P is a σ_i-group. And let H be a group of minimal order from $f(\sigma_i) \setminus h(\sigma_i)$. Since the formation $f(\sigma_i)$ is τ-closed, H is a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group. Let Q be the monolith of H. Then $Q = H^{h(\sigma_i)} = H^{\mathfrak{G}_{\sigma_i} \mathfrak{M}}$. Let us show that Q is not contained in $\Phi(H)$. Suppose the contrary. Then since $H / Q \in h(\sigma_i)$, we have $H / \Phi(H) \in h(\sigma_i)$. By the condition of the lemma the formation $\mathfrak{M}$ is saturated, and hence local. Therefore, by lemma 12 the formation $h(\sigma_i) = \mathfrak{G}_{\sigma_i} \mathfrak{M}$ is local, and hence $h(\sigma_i)$ is saturated. Consequently, $H \in h(\sigma_i)$. The obtained contradiction shows that $\Phi(H) = 1$.

Since $h(\sigma_i) = \mathfrak{G}_{\sigma_i} \mathfrak{M}$, we have $O_{\sigma_i}(H) = 1$. Consequently, by lemma 13 there exists a faithful irreducible $F_p H$-module P, where $p \in \sigma_i$. Let $F = P \rtimes H$. Then $P = C_F(P)$ is the minimal normal σ_i-subgroup of F. Since the σ-local definition f is integrated, $F \in \mathfrak{F}$ by lemma 7. It is clear that $O_{\sigma_{i'}, \sigma_i}(F) = O_{\sigma_i}(F)$. Moreover, since $O_{\sigma_i}(H) = 1$, we have

$$O_{\sigma_i}(F) = O_{\sigma_i}(F) \cap F = O_{\sigma_i}(F) \cap PH = P\left(O_{\sigma_i}(F) \cap H\right) = P.$$

Denote by h_1 the smallest τ-valued σ-local definition of $\mathfrak{H}_1 = l_\sigma^\tau \operatorname{form} F$. By lemma 5 we have $h_1(\sigma_i) = \tau\operatorname{form}\left(F / O_{\sigma_{i'},\sigma_i}(F)\right) = \tau\operatorname{form}(F/P) = \tau\operatorname{form} H$. If $\mathfrak{H}_1 \subset \mathfrak{F}$, then by condition $\mathfrak{H}_1 \subseteq \mathfrak{H}$. Hence, by lemma 6 we have the embedding $h_1 \leq h$. The latter contradicts the definition of the group H. Thus, $\mathfrak{H}_1 = \mathfrak{F}$, i. e. $\mathfrak{F} = l_\sigma^\tau \operatorname{form} F$.

Suppose that $\sigma_i \in \sigma(Q)$. Since $O_{\sigma_i}(H) = 1$, Q is a non-σ-primary group. It is clear that $H \in \mathfrak{F}$ and $O_{\sigma_{i'},\sigma_i}(H) = 1$. Suppose that the formation $\mathfrak{H}_2 = l_\sigma^\tau \operatorname{form} H \neq \mathfrak{F}$. Then by condition $\mathfrak{H}_2 \subseteq \mathfrak{H}$. Hence, $H \simeq H / O_{\sigma_{i'},\sigma_i}(H) \in h(\sigma_i)$, which contradicts the definition of the group H. Consequently, $\mathfrak{H}_2 = \mathfrak{F}$. Moreover, repeating the above arguments, we obtain that H is a minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{M}\right)$-group and $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$. Thus, we arrive at the situation already considered. Therefore, $\sigma_i \notin \sigma(Q)$.

Sufficiency. By lemma 11, to prove that $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation, it suffices to establish that $f(\sigma_t)$ is a minimal τ-closed non-$h(\sigma_t)$-formation for all $\sigma_t \in \sigma(P)$.

Let P be a non-σ-primary group. Then $O_{\sigma_{t'},\sigma_t}(G) = 1$ for all $\sigma_t \in \sigma(P)$. Consequently, by lemma 5 for all such σ_t we have $f(\sigma_t) = \tau\operatorname{form} G$. By lemma 8 the formation $f(\sigma_i)$ is τ-irreducible and its maximal τ-closed subformation coincides with $\tau\operatorname{form}\left(\mathfrak{X} \cup \{G/P\}\right)$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of G. By condition G is a $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{M}\right)$-group and $P = G^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$. Hence, if $\mathfrak{X}_1 = \mathrm{S}_{\overline{\tau}}(G/P)$, then $\mathfrak{X}_1 \subseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}$. Therefore, by lemmas 14 and 24 we have

$$\tau\operatorname{form}\left(\mathfrak{X} \cup \{G/P\}\right) = \tau\operatorname{form}\left(\mathfrak{X} \cup \mathfrak{X}_1\right) = \operatorname{form}\left(\mathfrak{X} \cup \mathfrak{X}_1\right) \subseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M} \subseteq \mathfrak{H} = h(\sigma_t)$$

for all $\sigma_t \in \sigma(P) \setminus \{\sigma_i\}$. Since also $P = G^{\mathfrak{H}} = G^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$, it follows that $f(\sigma_t)$ is not contained in $h(\sigma_t)$ for all $\sigma_t \in \sigma(P)$. Thus, if P is a non-σ-primary group, then $f(\sigma_t)$ is a minimal τ-closed non-$h(\sigma_t)$-formation for all $\sigma_t \in \sigma(P)$.

Let P be a σ_i-group. Then $O_{\sigma_{i'},\sigma_i}(G) = O_{\sigma_i}(G)$. Moreover, since $O_{\sigma_i}(H) = 1$, we have

$$O_{\sigma_i}(G) = O_{\sigma_i}(G) \cap G = O_{\sigma_i}(G) \cap PH = P\left(O_{\sigma_i}(G) \cap H\right) = P.$$

By lemma 5 we have

$$f(\sigma_i) = \tau\operatorname{form}\left(G / O_{\sigma_{i'},\sigma_i}(G)\right) = \tau\operatorname{form} H.$$

Since H is monolithic and $\Phi(H) = 1$, by lemma 8 $\tau\operatorname{form}\left(\mathfrak{X} \cup \{H/Q\}\right)$ is the unique maximal τ-closed subformation of $f(\sigma_i)$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of the group H. Now, as above, one can show that

$$\tau\operatorname{form}\left(\mathfrak{X} \cup \{H/Q\}\right) \subseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}.$$

Thus, $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation. Hence, $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation by lemma 11. The lemma is proved.

Lemma 27. *Let $\mathfrak{M}$ be a non-empty Abelian formation, $\mathfrak{H} = \mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\mathfrak{M}$. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, such that $\sigma_i \in \sigma(P)$, and either $P = G^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$ is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{M}\right)$-group, or $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and H satisfies one of the following conditions:*

1) H is a monolithic $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{M}\right)$-group with monolith $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{M}} \not\subseteq \Phi(H)$, such that $\sigma_i \notin \sigma(Q)$;

2) H is a minimal non-$\mathfrak{M}$-group of one of the following types:

a) the quaternion group of order 8, if $2 \notin \sigma_i$;

b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$;

c) a cyclic q-group, $q \notin \sigma_i$.

P r o o f. Let f be the smallest τ-valued σ-local definition of $\mathfrak{F}$, h be the canonical σ-local definition of $\mathfrak{H}$.

Necessity. Let $\mathfrak{F}$ be an $\mathfrak{H}_\sigma^\tau$-critical formation. Suppose that $\mathfrak{F}$ is not contained in $\left(\mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\right)\mathfrak{N} = \mathfrak{H}_1$, where $\mathfrak{N}$ is the class of all nilpotent groups. Since $\mathfrak{M} \subseteq \mathfrak{N}$, we have $\left(\mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\right)\mathfrak{M} \subseteq \left(\mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\right)\mathfrak{N} = \mathfrak{H}_1$. Hence, in this case $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}_1$-formation. Consequently, by lemma 26 we have that $\mathfrak{F}$ is a τ-closed σ-local formation generated by a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}_1$-group G with monolith $P = G^{\mathfrak{H}_1}$, such that $\sigma_i \in \sigma(P)$, and either P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{N}\right)$-group, and $P = G^{\mathfrak{G}_{\sigma_i}\mathfrak{N}}$, or $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and H is a monolithic $\overline{\tau}$-minimal non-$\left(\mathfrak{G}_{\sigma_i}\mathfrak{N}\right)$-group with monolith $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{N}}$, such that $Q \not\subseteq \Phi(H)$ and $\sigma_i \notin \sigma(Q)$.

Let P be a non-σ-primary group. Then by lemma 25 the unique maximal τ-closed σ-local subformation $\mathfrak{M}_1$ of $\mathfrak{F}$ has an integrated τ-valued σ-local definition m such that $m(\sigma_i) = \tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\})$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of the group G. By lemma 1 (4), (5) we have $m \le t$, where t is the canonical σ-local definition of $\mathfrak{M}_1$. Since by condition $\mathfrak{M}_1 \subseteq \mathfrak{H}$, by lemma 15 the embedding $m \le t \le h$ holds. By lemma 24 we have $h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}$. Hence, $(\mathfrak{X} \cup \{G/P\}) \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{M}$. Thus, if P is a non-σ-primary group, then $P = G^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$ and G is a $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i}\mathfrak{M})$-group.

Let P be a σ-primary group, $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of H. Then, taking into account lemma 23 and reasoning as in the previous paragraph, we obtain that $\mathfrak{X} \cup \{H/Q\} \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{M}$. Hence, H is a $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i}\mathfrak{M})$-group. Since also $H/Q \in \mathfrak{G}_{\sigma_i}\mathfrak{M}$, we have $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$. Thus, H satisfies condition 1).

Note that in both considered cases we have $P = G^{\mathfrak{H}}$. Indeed, if P is non-σ-primary, then since $G/P \in \mathfrak{G}_{\sigma_i}\mathfrak{M} \subseteq \mathfrak{H}$ we have $G/P \in \mathfrak{H}$. If P is σ-primary, then since $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$ and Q is a $\sigma_{i'}$-group, we again obtain $G/P \simeq H \in \mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\mathfrak{M} = \mathfrak{H}$. Assume that $G \in \mathfrak{H}$. Then $G \in \mathfrak{H}_1$, but $P = G^{\mathfrak{H}_1}$. Contradiction. Consequently, $P = G^{\mathfrak{H}}$.

Thus, $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}\,G$, where G is a group satisfying the condition of the lemma.

Consider the case when $\mathfrak{F} \subseteq \mathfrak{H}_1$. By lemma 11 $\mathfrak{F}$ is a τ-closed σ-local formation generated by a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group F with monolith $F^{\mathfrak{H}}$, such that for all $\sigma_j \in \sigma(F^{\mathfrak{H}})$ it holds that $f(\sigma_j)$ is a minimal τ-closed non-$h(\sigma_j)$-formation. Since $F/F^{\mathfrak{H}} \in \mathfrak{H} = \mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\mathfrak{M}$, we have $\sigma_i \in \sigma(F^{\mathfrak{H}})$. By lemma 24 the formation $\mathfrak{G}_{\sigma_i}\mathfrak{N}$ is the value at σ_i of the canonical σ-local definition of $\mathfrak{H}_1$. Since by our assumption $\mathfrak{F} \subseteq \mathfrak{H}_1$, we have $f(\sigma_i) \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{N}$. By lemma 5 (3) we have $f(\sigma_i) = \tau\,\mathrm{form}(A \mid A \in f(\sigma_i), O_{\sigma_i}(A) = 1)$. Hence, $f(\sigma_i) \subseteq \mathfrak{N}_{\sigma_{i'}}$, where $\mathfrak{N}_{\sigma_{i'}}$ is the formation of all nilpotent $\sigma_{i'}$-groups.

By lemma 16 all proper subformations of $f(\sigma_i)$ are hereditary. From this and from the fact that every proper τ-closed subformation of $f(\sigma_i)$ is contained in $h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}$, it follows that every proper subformation of $f(\sigma_i)$ is contained in $\mathfrak{M}$. On the other hand, it is clear that $f(\sigma_i)$ is not contained in $\mathfrak{M}$. Hence, $f(\sigma_i)$ is a minimal τ-closed non-$\mathfrak{M}$-formation.

Let $f(\sigma_i)$ not be contained in $\mathfrak{A}$. Then since $\mathfrak{M} \subseteq \mathfrak{A}$, it follows that $f(\sigma_i)$ is a minimal non-Abelian formation. By lemma 17 $f(\sigma_i)$ is the formation generated by H, where H is either the quaternion group of order 8 or an extraspecial group of order q^3 of prime odd exponent q. Since $f(\sigma_i) \subseteq \mathfrak{N}_{\sigma_{i'}}$, in the first case $2 \notin \sigma_i$, and in the second case $q \notin \sigma_i$.

Since in both these cases the group H is monolithic and $O_p(H) = 1$, where $p \in \sigma_i$, by lemma 13 there exists a faithful irreducible F_pH-module P. Let $G = P \rtimes H$. Then

$$O_{\sigma_{i'},\sigma_i}(G) = O_{\sigma_i}(G) = O_{\sigma_i}(G) \cap PH = P(O_{\sigma_i}(G) \cap H) = P,$$

since H is a $\sigma_{i'}$-group. Hence $G/O_{\sigma_i}(G) \simeq H \in f(\sigma_i)$. Therefore, $G \in \mathfrak{F}$ by lemma 7. If the τ-closed σ-local formation $\mathfrak{F}_1$, generated by the group G, is different from $\mathfrak{F}$, then by condition $\mathfrak{F}_1 \subseteq \mathfrak{H}$. But then $G/O_{\sigma_{i'},\sigma_i}(G) \simeq H \in h(\sigma_i)$. Consequently, $f(\sigma_i) \subseteq h(\sigma_i)$. Contradiction. Hence, $\mathfrak{F}_1 = \mathfrak{F}$. Moreover, obviously, $P = C_G(P) = G^{\mathfrak{H}}$. Since $f(\sigma_i)$ is nilpotent, we have $\mathrm{form}\,\mathfrak{X} \subseteq \mathrm{form}\,H = f(\sigma_i)$, where $\mathfrak{X}$ is the set of all proper subgroups of H. Since all proper subgroups of H are Abelian and the formation $f(\sigma_i)$ is non-Abelian, we have $\mathrm{form}\,\mathfrak{X} \subset f(\sigma_i)$. Hence, since $f(\sigma_i)$ is a minimal non-$\mathfrak{M}$-formation, H is a minimal non-$\mathfrak{M}$-group.

Let $f(\sigma_i) \subseteq \mathfrak{A}$ and H be a group of minimal order from $f(\sigma_i) \setminus \mathfrak{M}$. Since $f(\sigma_i)$ is hereditary, H is a minimal non-$\mathfrak{M}$-group. Moreover, because $f(\sigma_i)$ is a minimal non-$\mathfrak{M}$-formation, $f(\sigma_i)$ is the formation generated by the group H. Since the formation $f(\sigma_i)$ is Abelian, H is a cyclic q-group, where $q \in \sigma_j \in \sigma \setminus \{\sigma_i\}$.

Considering that $f(\sigma_i) \subseteq \mathfrak{N}_{\sigma_{i'}}$, then $O_p(H) = 1$, where $p \in \sigma_i$. Therefore, by lemma 13 there exists a simple faithful F_pH-module P. Let $G = P \rtimes H$. Since H does not belong to $\mathfrak{M}$, then, as above, one can show that $\mathfrak{F} = l_\sigma^\tau\,\mathrm{form}\,G$. Moreover, it is clear that $P = C_G(P) = G^{\mathfrak{H}}$ and H satisfy condition 2).

Sufficiency. By lemma 11, to prove that $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation, it suffices to show that $f(\sigma_r)$ is a minimal τ-closed non-$h(\sigma_r)$-formation for all $\sigma_r \in \sigma(P)$.

Let P be non-σ-primary. Then $O_{\sigma_{r'},\sigma_r}(G) = 1$ for every $\sigma_r \in \sigma(P)$. Consequently, $f(\sigma_r) = \tau\,\mathrm{form}\,G$. By lemma 8 the formation $f(\sigma_r)$ has a unique maximal τ-closed subformation, coinciding with $\tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\})$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of the group G. But by condition G is a $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i}\mathfrak{M})$-group and $P = G^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$. Hence, $\mathfrak{X} \cup \{G/P\} \subseteq \mathfrak{G}_{\sigma_i}\mathfrak{M} = h(\sigma_i)$. Consequently, the inclusion $\mathrm{form}(\mathfrak{X} \cup \{G/P\}) \subseteq h(\sigma_i)$ holds. But $\mathrm{form}(\mathfrak{X} \cup \{G/P\}) = \tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\})$. Indeed, if $L/P \in \overline{\tau}(G/P)$ and $L \neq G$, then $L \in \mathfrak{X}$. Therefore, $L/P \in \mathrm{H}(\mathfrak{X} \cup \{G/P\})$. Hence,

$$\mathrm{S}_{\overline{\tau}}(\mathfrak{X} \cup \{G/P\}) \subseteq \mathrm{H}(\mathfrak{X} \cup \{G/P\}).$$

But by lemma 18 we have

$$\tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\}) = \mathrm{HR}_0\mathrm{S}_{\overline{\tau}}(\mathfrak{X} \cup \{G/P\}) = \mathrm{form}(\mathrm{S}_{\overline{\tau}}(\mathfrak{X} \cup \{G/P\})) =$$
$$= \mathrm{form}(\mathrm{H}(\mathfrak{X} \cup \{G/P\})) = \mathrm{form}(\mathfrak{X} \cup \{G/P\}).$$

Thus,

$$\tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\}) \subseteq h(\sigma_i).$$

On the other hand, since by lemma 23 for any $\sigma_t \in \sigma \setminus \{\sigma_i\}$ we have $h(\sigma_t) = \mathfrak{G}_{\sigma_{i'}}\mathfrak{G}_{\sigma_i}\mathfrak{M}$, it follows that for all such σ_t the inclusion $\tau\,\mathrm{form}(\mathfrak{X} \cup \{G/P\}) \subseteq h(\sigma_t)$ also holds. Since $P = G^{\mathfrak{H}}$, it follows that $f(\sigma_t)$ is not contained in $h(\sigma_t)$ for all $\sigma_t \in \sigma(P)$. Thus, if P is a non-σ-primary group, then $f(\sigma_t)$ is a minimal τ-closed non-$h(\sigma_t)$-formation for all $\sigma_t \in \sigma(G^{\mathfrak{H}})$. Consequently, $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation.

Now let P be a σ_i-group. Then $O_{\sigma_{i'},\sigma_i}(G) = O_{\sigma_i}(G)$ and since $O_{\sigma_i}(H) = 1$ we have

$$O_{\sigma_i}(G) = O_{\sigma_i}(G) \cap G = O_{\sigma_i}(G) \cap PH = P(O_{\sigma_i}(G) \cap H) = P.$$

Consequently, $f(\sigma_i) = \tau\,\mathrm{form}(G/O_{\sigma_{i'},\sigma_i}(G)) = \tau\,\mathrm{form}\,H$ by lemma 5. It is clear that $f(\sigma_i)$ is not contained in $h(\sigma_i)$.

Let H satisfy condition 1). Then by lemma 8 the formation $f(\sigma_i)$ has a unique maximal τ-closed subformation $\mathfrak{M}_1 = \tau\,\mathrm{form}(\mathfrak{X} \cup \{H/Q\})$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of H. By condition H is a $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i}\mathfrak{M})$-group and $Q = H^{\mathfrak{G}_{\sigma_i}\mathfrak{M}}$. Hence,

$$\tau\,\mathrm{form}(\mathfrak{X} \cup \{H/Q\}) = \mathrm{form}(\mathfrak{X} \cup \{H/Q\}) \subseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{M}.$$

It is clear that $f(\sigma_i)$ is not contained in $h(\sigma_i)$. Thus, $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation, and hence $\mathfrak{F}$ is an $\mathfrak{H}_\sigma^\tau$-critical formation.

Now let H satisfy condition 2) and let H be the quaternion group of order 8, $2 \notin \sigma_i$. And let $\mathfrak{M}_1$ be the formation of Abelian groups of exponent dividing 4. Suppose $\mathfrak{M}_1 \not\subseteq f(\sigma_i)$ and let K be a group of minimal order from $\mathfrak{M}_1 \setminus f(\sigma_i)$. Then K is monolithic. Consequently, K is a cyclic primary group. Thus, either $|K| = 4$, or $|K| = 2$. But the quaternion group of order 8 has cyclic subgroups of orders 2 and 4. Consequently, $K \in f(\sigma_i)$. The obtained contradiction shows that $\mathfrak{M}_1 \subseteq f(\sigma_i)$. Let $\mathfrak{H}_1$ be an arbitrary proper subformation of $f(\sigma_i)$. Then by lemma 17 we have $\mathfrak{H}_1 \subseteq \mathfrak{A}$. Assume that $\mathfrak{H}_1 \not\subseteq \mathfrak{M}_1$. Let M be a group of minimal order from $\mathfrak{H}_1 \setminus \mathfrak{M}$. Then M is monolithic. Since the group M is Abelian, it is cyclic. By lemma 19 the exponent of M divides the exponent of H. Thus, the exponent of M is 2 or 4. But then, obviously, $M \in \mathfrak{M}_1$. The obtained contradiction shows that every proper subformation of $f(\sigma_i)$ is contained in $\mathfrak{M}_1$. But, obviously, $\mathfrak{M}_1 = \mathrm{form}(Z_4)$, where Z_4 is the cyclic group of order 4. Hence, $\mathfrak{M}_1 \subseteq \tau\,\mathrm{form}\,\mathfrak{X}$, where $\mathfrak{X}$ is the set of all proper subgroups of H. Thus,

$$\mathfrak{M}_1 = \mathrm{form}(Z_4) = \tau\,\mathrm{form}\,\mathfrak{X}.$$

Since by condition H is a minimal non-$\mathfrak{M}$-group, we have $\mathfrak{M}_1 \subseteq \mathfrak{M}$. Consequently, $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation.

Now let H be an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$. Then by lemma 17 $\tau\,\mathrm{form}\,H = f(\sigma_i)$ is a minimal τ-closed non-Abelian formation. Let M be a maximal subformation of $f(\sigma_i)$. Then M is an Abelian formation and every non-identity group from M has exponent q. Since K is a minimal non-$h(\sigma_i)$-group, $h(\sigma_i)$ contains groups of order q. But then $M \subseteq h(\sigma_i)$. Consequently, $f(\sigma_i)$ is a minimal non-$h(\sigma_i)$-formation.

Finally, let H be a cyclic q-group, $q \notin \sigma_i$. Let $|H| = q^n$ and M be a maximal subgroup of H. Then $M \in h(\sigma_i)$, but $H \notin h(\sigma_i)$. Hence, $f(\sigma_i) \not\subseteq h(\sigma_i)$ and $\mathrm{form}\,M \subseteq h(\sigma_i)$. Let us show that every proper subformation of

$f(\sigma_i)$ is contained in $\mathrm{form}\,M$. Suppose that in $f(\sigma_i)$ there is a proper subformation $\mathfrak{M}$ such that $\mathfrak{M} \not\subseteq \mathrm{form}\,M$, and let A be a group of minimal order from $\mathfrak{M} \setminus \mathrm{form}\,M$. Then A is monolithic. Since $f(\sigma_i)$ is Abelian, A is a cyclic group of order q^m. According to lemma 19 the exponent q^m of A divides the exponent q^n of H. If $m = n$, then $A \simeq H$ and then $f(\sigma_i) \subseteq \mathrm{form}\,A \subseteq \mathfrak{M} \subset f(\sigma_i)$. The obtained contradiction shows that $m \neq n$. But the order of M is q^{n-1}. Hence, either A is isomorphic to some proper subgroup of M, or $A \simeq M$. This means that $A \in \mathrm{form}\,M$. Contradiction. Consequently, $f(\sigma_i)$ is a minimal non-$h(\sigma_i)$-formation.

Thus, in all considered cases $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation. Therefore, $\mathfrak{F}$ is an $\mathfrak{H}^{\tau}_{\sigma}$-critical formation by lemma 11. The lemma is proved.

Lemma 28. *Let h be the canonical definition of* $\mathfrak{H}$ *and* $G = P \rtimes K$ *be a monolithic group with monolith* $P = C_G(P) = G^{\mathfrak{H}}$ *– a p-group,* $p \in \sigma_i \in \sigma$, $O_{\sigma_i}(K) = 1$. *Then G is a* $\overline{\tau}$*-minimal non-*$\mathfrak{H}$*-group if and only if K is a* $\overline{\tau}$*-minimal non-*$h(\sigma_i)$*-group.*

P r o o f. *Sufficiency.* Let K be a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group and $M \in \overline{\tau}(G) \setminus \{G\}$. Let us show that $M \in \mathfrak{H}$. If $G = PM$, then, obviously, $M \cap P = 1$. Hence, $M \simeq M / P \cap M \simeq MP / P = G / P \in \mathfrak{H}$. Let $PM \neq G$. Then $PM / P \in \overline{\tau}(G / P) \setminus \{G / P\}$. But $G / P \simeq K$. Consequently, in K there exists a proper $\overline{\tau}$-subgroup D such that $MP / P \simeq D \in h(\sigma_i)$. So,

$$M / M \cap P \simeq MP / P \in h(\sigma_i) = \mathfrak{G}_{\sigma_i} h(\sigma_i).$$

Hence, $M \in h(\sigma_i)$. Thus, every proper $\overline{\tau}$-subgroup of G belongs to $\mathfrak{H}$. Since by condition $P = G^{\mathfrak{H}}$, it follows that G is a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group.

Necessity. Let G be a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group and $M \in \overline{\tau}(K) \setminus \{K\}$. If under the isomorphism $G / P \simeq K$ the subgroup M corresponds to the subgroup D / P in the group G / P, then $D \in \overline{\tau}(G) \setminus \{G\}$. Consequently, $D \in \mathfrak{H}$. Therefore, $D / O_{\sigma_i', \sigma_i}(D) \in h(\sigma_i)$. Then since

$$D = D \cap G = D \cap PK = P \rtimes (D \cap K),\ P = C_G(P) \text{ and } O_{\sigma_i}(K) = 1,$$

we have $O_{\sigma_i', \sigma_i}(D) = O_{\sigma_i}(D)$. It is also clear that $O_{\sigma_i}(D) / P = O_{\sigma_i}(D / P)$. Consequently, since $D / O_{\sigma_i', \sigma_i}(D) \in h(\sigma_i)$, it follows that $M / O_{\sigma_i}(M) \in h(\sigma_i) = \mathfrak{G}_{\sigma_i} h(\sigma_i)$. Hence, $M \in h(\sigma_i)$. Thus, every proper $\overline{\tau}$-subgroup of K belongs to $h(\sigma_i)$. If, moreover, $K \in h(\sigma_i)$, then $G \in \mathfrak{H}$ by lemma 7, which is impossible. Therefore, K is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group. The lemma is proved.

Proof of theorem 1

Necessity. According to the condition, the formation $\mathfrak{H}$ has a σ-local definition f all of whose non-Abelian values are σ-local. By lemma 20 we can assume that such a σ-local definition is integrated. By lemma 1 (2) we have

$$\mathfrak{H} = \left(\cap_{\sigma_i \in \sigma(\mathfrak{H})} \mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{G}_{\sigma(\mathfrak{H})}.$$

Since by condition every proper τ-closed σ-local subformation of $\mathfrak{F}$ is contained in $\mathfrak{H}$, but $\mathfrak{F} \not\subseteq \mathfrak{H}$, then either $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{G}_{\sigma(\mathfrak{H})}$-formation, or there exists $\sigma_i \in \sigma(\mathfrak{H})$ such that $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\left(\mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right)$-formation.

Consider first the case when $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{G}_{\sigma(\mathfrak{H})}$-formation. Then by lemma 21 we have $\mathfrak{F} = \mathfrak{G}_{\sigma_i} = l^{\tau}_{\sigma}\, \mathrm{form}\, G$, where G is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) \subseteq \{1, G\}$. It is clear that G is a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group and $G = G^{\mathfrak{H}}$. Thus, the group G satisfies condition 1) of the theorem.

Now let $\sigma_i \in \sigma(\mathfrak{H})$ be such that $\mathfrak{F}$ is a τ-closed σ-local non-$\left(\mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right)$-formation.

Assume first that $f(\sigma_i)$ is a σ-local formation. By lemma 1 (5) $h(\sigma_j) = \mathfrak{G}_{\sigma_j} f(\sigma_j)$ for any σ_j. Since by lemma 1 (2) the formation $f(\sigma_i)$ is saturated, by lemma 26 we have $\mathfrak{F} = l^{\tau}_{\sigma}\, \mathrm{form}\, G$, where G is a monolithic group with monolith $P = G^{\mathfrak{G}_{\sigma_i'} h(\sigma_i)}$, such that $\sigma_i \in \sigma(P)$ and either P is a non-σ-primary group, G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group and $P = G^{h(\sigma_i)}$, or $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with monolith $Q = K^{h(\sigma_i)}$, such that $Q \not\subseteq \Phi(K)$ and $\sigma_i \notin \sigma(Q)$.

Denote by $\mathfrak{M}$ the maximal τ-closed σ-local subformation of $\mathfrak{F}$. Consider the case when P is a non-σ-primary group. By lemma 25 the formation $\mathfrak{M}$ has an integrated σ-local definition m such that $m(\sigma_j) = \tau\operatorname{form}(\mathfrak{X} \cup \{G/P\})$ for all $\sigma_j \in \sigma(P)$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of G. By lemma 15 we have $m \leq h$. Thus, for all $\sigma_j \in \sigma(P)$ it holds that all proper $\overline{\tau}$-subgroups of G belong to $h(\sigma_j)$ and $G/P \in h(\sigma_j)$. Suppose that there exists $\sigma_j \in \sigma(P)$ such that $G \in h(\sigma_j)$. Then since the σ-local definition h is integrated, we have

$$G \in h(\sigma_j) \subseteq \mathfrak{H} \subseteq \mathfrak{G}_{\sigma_{i'}} \mathfrak{G}_{\sigma_i} f(\sigma_i).$$

But $P = G^{\mathfrak{G}_{\sigma_{i'}} h(\sigma_i)} = G^{\mathfrak{G}_{\sigma_{i'}} \mathfrak{G}_{\sigma_i} f(\sigma_i)}$. The obtained contradiction shows that G is a $\overline{\tau}$-minimal non-$h(\sigma_j)$-group for all $\sigma_j \in \sigma(P)$. It is clear that $P = G^{\mathfrak{H}}$. Thus, in this case $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, satisfying condition 2).

Let P be a σ-primary group. By lemma 28 the group G is a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group. By lemma 23 the formation $\mathfrak{M}$ has an integrated definition m such that $m(\sigma_i) = \tau\operatorname{form}(\mathfrak{X} \cup \{K/Q\})$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of K, $m(\sigma_j) = \tau\operatorname{form}(G/O_{\sigma_{j'},\sigma_j}(G))$ for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$ and $m(\sigma_j) = \varnothing$ for all $\sigma_j \notin \sigma(G)$. By condition $\mathfrak{M} \subseteq \mathfrak{H}$. Hence, $m \leq h$. Since $m(\sigma_j) = \tau\operatorname{form}(G/O_{\sigma_{j'},\sigma_j}(G))$ for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$, it follows that for all such σ_j we have $G/O_{\sigma_{j'},\sigma_j}(G) \in h(\sigma_j)$. Let us show that $G/O_{\sigma_{j'},\sigma_j}(G) \simeq K/O_{\sigma_{j'},\sigma_j}(K)$. Indeed, since $\sigma_i \neq \sigma_j$, we have $O_{\sigma_{j'},\sigma_j}(G)/P = O_{\sigma_{j'},\sigma_j}(G/P)$. Consequently,

$$K/O_{\sigma_{j'},\sigma_j}(K) \simeq (G/P)/O_{\sigma_{j'},\sigma_j}(G/P) = (G/P)/\left(O_{\sigma_{j'},\sigma_j}(G)/P\right) \simeq G/O_{\sigma_{j'},\sigma_j}(G).$$

Therefore, for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$ we have $K/O_{\sigma_{j'},\sigma_j}(K) \in h(\sigma_j)$. Let us show that $P = G^{\mathfrak{H}}$. It is clear that $G \notin \mathfrak{H}$. If Q is a non-σ-primary group, then $O_{\sigma_{j'},\sigma_j}(K) = 1$. Consequently, $G/P \simeq K \in h(\sigma_j) \subseteq \mathfrak{H}$. If Q is a σ_j-group, then $O_{\sigma_{j'},\sigma_j}(K) = O_{\sigma_j}(K)$. Hence, $K/O_{\sigma_{j'},\sigma_j}(K) = K/O_{\sigma_j}(K) \in h(\sigma_j) = \mathfrak{G}_{\sigma_j} h(\sigma_j)$. Consequently,

$$G/P \simeq K \in \mathfrak{G}_{\sigma_j} h(\sigma_j) = h(\sigma_j) \subseteq \mathfrak{H}.$$

Thus, $P = G^{\mathfrak{H}}$ and G satisfies condition 3).

Consider now the case when $f(\sigma_i)$ is an Abelian formation. Applying lemma 27, we see that $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_{i'}} \mathfrak{G}_{\sigma_i} f(\sigma_i))$-group with monolith $P = G^{\mathfrak{H}}$, such that $\sigma_i \in \sigma(P)$, and either $P = G^{\mathfrak{G}_{\sigma_i} f(\sigma_i)}$ is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i} f(\sigma_i))$-group, or $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and H satisfies one of the following conditions: 1) H is a monolithic $\overline{\tau}$-minimal non-$(\mathfrak{G}_{\sigma_i} f(\sigma_i))$-group with monolith $Q = H^{\mathfrak{G}_{\sigma_i} f(\sigma_i)} \not\subseteq \Phi(H)$, such that $\sigma_i \notin \sigma(Q)$; 2) H is a minimal non-$f(\sigma_i)$-group of one of the following types: a) the quaternion group of order 8, if $2 \notin \sigma_i$; b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$; c) a cyclic q-group, $q \notin \sigma_i$.

Let P be a non-σ-primary group. By lemma 25 the formation $\mathfrak{M}$ has an integrated σ-local definition m such that $m(\sigma_j) = \tau\operatorname{form}(\mathfrak{X} \cup \{G/P\})$ for all $\sigma_j \in \sigma(P)$. By condition $\mathfrak{M} \subseteq \mathfrak{H}$. Consequently, $m \leq h$. Thus, for all $\sigma_j \in \sigma(P)$ it holds that all proper $\overline{\tau}$-subgroups of G belong to $h(\sigma_j)$ and $G/P \in h(\sigma_j)$. Suppose that there exists $\sigma_j \in \sigma(P)$ such that $G \in h(\sigma_j)$. Then since the σ-local definition h is integrated, we have $G \in \mathfrak{H} \subseteq \mathfrak{G}_{\sigma_{i'}} h(\sigma_i)$. The obtained contradiction shows that $P = G^{h(\sigma_j)}$ for all $\sigma_j \in \sigma(P)$. It is also clear that $P = G^{\mathfrak{H}}$. Thus, in this case $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, satisfying condition 2).

Let P be a σ_i-group. Then G is a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group by lemma 28.

Let H satisfy condition 1). Then by lemma 23 the formation $\mathfrak{M}$ has an integrated definition m such that $m(\sigma_i) = \tau\operatorname{form}(\mathfrak{X} \cup \{H/Q\})$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of the group H. By condition $\mathfrak{M} \subseteq \mathfrak{H}$. Hence, $m \leq h$. Consequently, $\mathfrak{X} \cup \{H/Q\} \subseteq h(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$. If $H \in h(\sigma_i)$, then $G/P \simeq H \in h(\sigma_i)$. Hence,

$$G \in \mathfrak{G}_{\sigma_i} h(\sigma_i) = h(\sigma_i) \subseteq \mathfrak{H}.$$

The obtained contradiction shows that H is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group and $H/Q \in h(\sigma_i)$. Hence, $Q = H^{h(\sigma_i)}$ and H satisfies condition 3) of the theorem.

If H satisfies condition 2), then, obviously, H satisfies condition 3) of the theorem.

Let us show that $P = G^{\mathfrak{H}}$. Obviously, $G \notin \mathfrak{H}$. Suppose that $G / P \notin \mathfrak{H}$. Then $H \notin \mathfrak{H}$. Consequently, $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G = l_\sigma^\tau \operatorname{form} H$. Applying now lemma 5, we see that

$$f(\sigma_i) = \tau \operatorname{form}\left(G / O_{\sigma_{i'}, \sigma_i}(G)\right) = \tau \operatorname{form}(G / P) = \tau \operatorname{form} H = \tau \operatorname{form}\left(H / O_{\sigma_{i'}, \sigma_i}(H)\right).$$

If the group H satisfies condition 1), then by lemma 8 the formation $\tau \operatorname{form}\left(\mathfrak{X} \cup \{H / Q\}\right)$ is the unique maximal τ-closed subformation of $\tau \operatorname{form} H$, where $\mathfrak{X}$ is the set of all proper $\overline{\tau}$-subgroups of H. Since Q is a $\sigma_{i'}$-group, we have $Q \subseteq O_{\sigma_{i'}, \sigma_i}(H)$. Hence, $H / O_{\sigma_{i'}, \sigma_i}(K) \in \tau \operatorname{form}\left(\mathfrak{X} \cup \{H / Q\}\right)$. Thus,

$$H / O_{\sigma_{i'}, \sigma_i}(H) \in \tau \operatorname{form}\left(\mathfrak{X} \cup \{H / Q\}\right) \subset \tau \operatorname{form} H = \tau \operatorname{form}\left(H / O_{\sigma_{i'}, \sigma_i}(H)\right).$$

Contradiction.

Now let H satisfy one of conditions 2). Then H is a σ-primary group, i. e. a σ_i-group for some $\sigma_i \in \sigma$. Consequently, $H = O_{\sigma_i}(H) = O_{\sigma_{i'}, \sigma_i}(H)$. Hence, $f(\sigma_i) = \tau \operatorname{form} H = \tau \operatorname{form}\left(H / O_{\sigma_{i'}, \sigma_i}(H)\right) = (1)$. The obtained contradiction shows that $G / P \in \mathfrak{H}$. Therefore, $P = G^{\mathfrak{H}}$.

Sufficiency. Let f be the smallest τ-valued σ-local definition of $\mathfrak{F}$. To prove sufficiency, by lemma 11 it suffices to establish that $f(\sigma_j)$ is a minimal τ-closed non-$h(\sigma_j)$-formation for every $\sigma_j \in \sigma(P)$.

If the group G satisfies condition 1), then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation by lemma 21.

Let G satisfy condition 2) and $\sigma_j \in \sigma(P)$. Then by lemma 5 we have

$$f(\sigma_j) = \tau \operatorname{form}\left(G / O_{\sigma_{j'}, \sigma_i}(G)\right) = \tau \operatorname{form} G.$$

Let $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of G. Then by condition $\mathfrak{X} \cup \{G / P\} \subseteq h(\sigma_j)$. But $\tau \operatorname{form}\left(\mathfrak{X} \cup \{G / P\}\right)$ is the unique maximal τ-closed subformation of $\tau \operatorname{form} G$ by lemma 8. It is also clear that G does not belong to $h(\sigma_j)$. Hence, $f(\sigma_j)$ is a minimal τ-closed non-$h(\sigma_j)$-formation.

Now let G satisfy condition 3) and $\Phi(K) = 1$. Then

$$f(\sigma_i) = \tau \operatorname{form}\left(G / O_{\sigma_{i'}, \sigma_i}(G)\right) = \tau \operatorname{form} K.$$

Let $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of K. Then by condition $\mathfrak{X} \subseteq h(\sigma_i)$ and $K / Q \in h(\sigma_i)$. Hence, $\tau \operatorname{form}\left(\mathfrak{X} \cup \{K / Q\}\right) \subseteq h(\sigma_i)$. By lemma 8 $\tau \operatorname{form}\left(\mathfrak{X} \cup \{K / Q\}\right)$ is the unique maximal τ-closed subformation of $f(\sigma_i) = \tau \operatorname{form} K$. Hence, every proper τ-closed subformation of $f(\sigma_i)$ is contained in $h(\sigma_i)$. If $f(\sigma_i) \subseteq h(\sigma_i)$, then $G / P \in h(\sigma_i)$. Hence, by lemma 7 we have $G \in \mathfrak{H}$. Contradiction. So, $f(\sigma_i) \not\subseteq h(\sigma_i)$. Thus, $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation.

Let K satisfy condition a), i. e. K is the quaternion group of order 8. Then by lemma 17 $f(\sigma_i) = \tau \operatorname{form} K$ is a minimal non-Abelian formation. Let $\mathfrak{M}$ be a maximal subformation of $f(\sigma_i)$. It is easy to see that $\mathfrak{M}$ is the formation of Abelian groups of exponent dividing 4, and $\mathfrak{M}$ is generated by any Abelian group of exponent 4. Among the subgroups of K there is a proper Abelian normal subgroup of exponent 4. Hence, $\mathfrak{M} \subseteq \operatorname{form} \mathfrak{X}$, where $\mathfrak{X}$ is the set of all proper subgroups of K; moreover, the subgroups of K are Abelian and their exponents divide 4. Consequently, $\operatorname{form} \mathfrak{X} \subseteq \mathfrak{M}$. Thus, $\mathfrak{M} = \operatorname{form} \mathfrak{X}$. But by condition $\mathfrak{X} \subseteq h(\sigma_i)$. Hence, $\mathfrak{M} \subseteq h(\sigma_i)$. So, $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation.

Now let K satisfy condition b), i. e. K is an extraspecial group of order q^3 of prime odd exponent q. Then by lemma 17 $\tau \operatorname{form} K = f(\sigma_i)$ is a minimal τ-closed non-Abelian formation. Let M be a maximal subformation of $f(\sigma_i)$. Then M is an Abelian formation and every non-identity group from M has exponent q. Since K is a minimal non-$h(\sigma_i)$-group, $h(\sigma_i)$ contains groups of order q. But then $M \subseteq h(\sigma_i)$. Consequently, $f(\sigma_i)$ is a minimal non-$h(\sigma_i)$-formation.

Finally, let K satisfy condition c), i. e. K is a cyclic q-group. Let $|K| = q^n$ and M be a maximal subgroup of K. Then $M \in h(\sigma_i)$, but $K \notin h(\sigma_i)$. Hence, $f(\sigma_i) \not\subseteq h(\sigma_i)$ and $\operatorname{form} M \subseteq h(\sigma_i)$. To complete the proof of the theorem it is necessary only to establish that every proper subformation of $f(\sigma_i)$ is contained in $\operatorname{form} M$. Suppose that in $f(\sigma_i)$ there is a proper subformation $\mathfrak{M}$ such that $\mathfrak{M} \not\subseteq \operatorname{form} M$, and let A be a group of minimal order from $\mathfrak{M} \setminus \operatorname{form} M$. Then A is monolithic. Since the formation $f(\sigma_i)$ is Abelian, A is a cyclic group of order q^m. According to lemma 19 the exponent q^m of A divides the exponent q^n of K. If $m = n$, then $A \simeq K$ and then $f(\sigma_i) \subseteq \operatorname{form} A \subseteq \mathfrak{M} \subset f(\sigma_i)$. The obtained contradiction shows that $m \neq n$. But the order of M is q^{n-1}. Hence, either A is isomorphic to some proper subgroup of M, or $A \simeq M$. This means that $A \in \operatorname{form} M$. Contradiction. Consequently, $f(\sigma_i)$ is a minimal non-$h(\sigma_i)$-formation.

By lemma 16 in each of the considered cases $f(\sigma_i)$ is a hereditary formation, therefore $f(\sigma_i)$ is a minimal τ-closed non-$h(\sigma_i)$-formation. Applying now lemma 11 we conclude that $\mathfrak{F}$ is an $\mathfrak{H}_\sigma^\tau$-critical formation. The theorem is proved.

Recall that a formation $\mathfrak{F}$ is called 2-multiply σ-local [10], if $\mathfrak{F}$ has a σ-local definition all of whose non-empty values are σ-local. It is clear that every 2-multiply σ-local formation is a σ-local formation of classical type.

Corollary 1. *Let $\mathfrak{H}$ be a 2-multiply σ-local formation and h be its canonical σ-local definition. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $G = P$ is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) \subseteq \{1, G\}$;

2) P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is a monolithic $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with monolith $Q = K^{h(\sigma_i)} \not\subseteq \Phi(K)$, where $\sigma_i \notin \sigma(Q)$.

Corollary 2. *Let $\mathfrak{H} = LF_\sigma(h)$, where h is an integrated σ-local definition all of whose non-empty values are Abelian. Then $\mathfrak{F}$ is a minimal τ-closed σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma^\tau \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $G = P$ is a simple σ_i-group such that $\sigma_i \notin \sigma(\mathfrak{H})$ and $\tau(G) \subseteq \{1, G\}$;

2) P is a non-σ-primary group and G is a $\overline{\tau}$-minimal non-$h(\sigma_i)$-group with $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is a minimal non-$h(\sigma_i)$-group of one of the following types:

a) the quaternion group of order 8, if $2 \notin \sigma_i$;

b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$;

c) a cyclic q-group, $q \notin \sigma_i$.

In the case when τ is the trivial subgroup functor, from theorem 1 we obtain following corollary.

Corollary 3 [7, theorem A]. *Let $\mathfrak{H}$ be a σ-local formation of classical type and h be its canonical σ-local definition. Then $\mathfrak{F}$ is a minimal σ-local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = l_\sigma \operatorname{form} G$, where G is a monolithic group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $G = P$ is a simple σ_i-group, $\sigma_i \notin \sigma(\mathfrak{H})$;

2) P is a non-σ-primary group and $P = G^{h(\sigma_i)}$ for all $\sigma_i \in \sigma(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, $p \in \sigma_i$, and K is either a monolithic group with monolith $Q = K^{h(\sigma_i)} \not\subseteq \Phi(K)$, where $\sigma_i \notin \sigma(Q)$, or a minimal non-$h(\sigma_i)$-group of one of the following types:

a) the quaternion group of order 8, if $2 \notin \sigma_i$;

b) an extraspecial group of order q^3 of prime odd exponent $q \notin \sigma_i$;

c) a cyclic q-group, $q \notin \sigma_i$.

In the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \{5\}, \ldots\}$ from theorem 1 we obtain following corollary.

Corollary 4 [4, theorem 2.2.10]. *Let $\mathfrak{H}$ be a formation of classical type and h be its canonical screen. Then $\mathfrak{F}$ is a minimal τ-closed local non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = \tau^l \operatorname{form} G$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $G = P$ is a group of prime order $p \notin \pi(\mathfrak{H})$;

2) P is a non-Abelian group and G is a $\overline{\tau}$-minimal non-$h(p)$-group with $P = G^{h(p)}$ for all $p \in \pi(P)$;

3) $G = P \rtimes H$, where $P = C_G(P)$ is a p-group, and H is either a monolithic $\overline{\tau}$-minimal non-$h(p)$-group with monolith $Q = H^{h(p)} \not\subseteq \Phi(H)$, where p does not divide $|Q|$, or a minimal non-$h(p)$-group of one of the following types:

a) the quaternion group of order 8;

b) a non-Abelian group of order q^3 of prime odd exponent q;

c) a cyclic primary group.

If, in addition, τ is the trivial subgroup functor, then we obtain following corollary.

Corollary 5 [3, theorem]. *Let $\mathfrak{H} = LF(h)$ be a formation of classical type and h be its maximal integrated local screen. The formation $\mathfrak{F}$ is an $\mathfrak{H}_l$-critical formation if and only if $\mathfrak{F} = l\,\mathrm{form}\,G$, where G is a monolithic group with monolith $P = G^{\mathfrak{H}}$, and one of the following conditions holds:*

1) $P = G$ is a group of prime order;

2) P is a non-Abelian group and $P = G^{h(p)}$ for all $p \in \pi(P)$;

3) $G = P \rtimes K$, where $P = C_G(P)$ is a p-group, and K is a monolithic group with monolith $Q = K^{h(p)}$ (p does not divide $|Q|$), such that either $\Phi(K) = 1$ and $K^{h(q)} \subseteq Q$ for all $q \in \pi(Q)$, or K is a minimal non-$h(p)$-group of one of the following types:

a) a cyclic primary group;

b) the quaternion group of order 8;

c) a non-Abelian group of order q^3 of prime odd exponent q.

References

1. Shemetkov LA. Screens of step formations. In: Chernikov SN, editor. *Proceedings of the 6th All-Union symposium on group theory; 1978 September 19–21; Cherkasy, Ukraine*. Kyiv: Naukova dumka; 1980. p. 37–50. Russian.

2. Skiba AN. On critical formations. *Izvestiya Akademii nauk Belorusskoi SSR. Seriya fiziko-matematicheskikh nauk*. 1980;4:27–33. Russian.

3. Skiba AN. On critical formations. In: Chernikov NS, editor. *Infinite groups and related algebraic structures*. Kyiv: Institute of Mathematics of the Academy of Sciences of Ukraine; 1993. p. 258–268. Russian.

4. Skiba AN. *Algebra of formations.* Minsk: Belaruskaja navuka; 1997. 240 p. Russian.

5. Shemetkov LA, Skiba AN. *Formations of algebraic systems*. Moscow: Nauka; 1989. 256 p. (Sovremennaya algebra). Russian.

6. Safonova IN. On minimal σ-local non-$\mathfrak{H}$-formations of finite groups. *Problems of Physics, Mathematics and Technics*. 2020;4:105–112. Russian. EDN: MCGNUX.

7. Safonova IN. On critical σ-local formations of finite groups. *Proceedings of the Institute of Mathematics of the National Academy of Sciences of Belarus.* 2023;31(2):63–80. Russian. EDN: NEPDVI.

8. Safonova IN, Skrundz VV. On $\mathfrak{H}_\sigma^\tau$-critical formations of finite groups. *Problems of Physics, Mathematics and Technics.* 2025;3:99–111. Russian. EDN: BFOPLO.

9. Skiba AN. On σ-subnormal and σ-permutable subgroups of finite groups. *Journal of Algebra*. 2015;436:1–16. DOI: 10.1016/j.jalgebra.2015.04.010.

10. Skiba AN. On one generalization of the local formations. *Problems of Physics, Mathematics and Technics.* 2018;1:79–82. Russian. EDN: YWUMMV.

11. Chi Z, Safonov VG, Skiba AN. On one application of the theory of *n*-multiply σ-local formations of finite groups. *Problems of Physics, Mathematics and Technics.* 2018;2:85–88. Russian. EDN: XUFGEH.

12. Zhang C, Skiba AN. On $\sum_t^\sigma$-closed classes of finite groups. *Ukrainian Mathematical Journal.* 2019;70(12):1966–1977. DOI: 10.1007/s11253-019-01619-6.

13. Chi Z, Safonov VG, Skiba AN. On *n*-multiply σ-local formations of finite groups. *Communications in Algebra*. 2019;47(3):957–968. DOI: 10.1080/00927872.2018.1498875.

14. Safonova IN, Safonov VG. On some properties of the lattice of totally σ-local formations of finite groups. *Journal of the Belarusian State University. Mathematics and Informatics*. 2020;3:6–16. DOI: 10.33581/2520-6508-2020-3-6-16.

15. Safonova IN. Some properties of *n*-multiply σ-local formations of finite groups. *Asian-European Journal of Mathematics*. 2022;15(7):2250138. DOI: 10.1142/S1793557122501388.

16. Safonova IN. On the τ-closedness of *n*-multiply σ-local formation. *Advances in Group Theory and Applications*. 2024;18:123–136. DOI: 10.32037/agta-2024-005.

17. Doerk K, Hawkes To. *Finite soluble groups.* Berlin: Walter de Gruyter; 1992. X, 891 p. (de Gruyter expositions in mathematics; volume 4). DOI: 10.1515/9783110870138.

18. Shemetkov LA. *Formations of finite groups.* Moscow: Nauka; 1978. 272 p. (Sovremennaya algebra). Russian.

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