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ПРЕДЕЛЬНАЯ ТЕОРЕМА О СХОДИМОСТИ К ЛОКАЛЬНОМУ ВРЕМЕНИ БРОУНОВСКОЙ ЭКСКУРСИИ

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Аннотация. Исследуется целочисленное случайное блуждание $\{S_i,\ i \geq 0\}$ с нулевым сносом и конечной дисперсией σ^2. Для случайного процесса, сопоставляющего переменной $u > 0$ число попаданий указанного блуждания в состояние $\lfloor u\sigma\sqrt{n} \rfloor$ до момента n и рассматриваемого при условии, что $S_1 > 0,\ \ldots,\ S_{n-1} > 0,\ S_n \leq 0$, доказывается функциональная предельная теорема о его сходимости к локальному времени броуновской экскурсии.

Ключевые слова: случайное блуждание; условное броуновское движение; локальное время условных броуновских движений; функциональная предельная теорема.

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LIMIT THEOREM ON CONVERGENCE TO THE LOCAL TIME OF A BROWNIAN EXCURSION

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Abstract. An integer random walk $\{S_i,\ i \geq 0\}$ with zero drift and finite variance σ^2 is investigated. For a random process that assigns, to a variable $u > 0$, the number of hits of the specified random walk into the state $\lfloor u\sigma\sqrt{n} \rfloor$ up to time n and is considered under the condition that $S_1 > 0,\ \ldots,\ S_{n-1} > 0,\ S_n \leq 0$, a functional limit theorem concerning convergence in distribution of the process to the local time of a Brownian excursion is proved.

Keywords: random walk; conditional Brownian motion; local time of conditional Brownian motions; functional limit theorem.

Statement of the problem and the main result

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We need the following assumptions about the distribution of the random variable X_1.

Assumption 1. The random variable X_1 has an arithmetic distribution with the maximum span 1, and $\mathbf{P}(X_1 = 0) > 0$.

Instead of the above assumption, we can consider a more general one: the random variable X_1 has a lattice distribution, i. e. its possible values belong to the lattice $\{a + kh,\ k \in \mathbf{Z}\}$, where $h > 0$ and $a \in [0,\ h)$, the maximum of all possible h is 1, and the value of a corresponding to this h is 0.

Assumption 2. The following equalities hold:

$$\mathbf{E}X_1 = 0,\ \mathbf{E}X_1^2 = \sigma^2,\ 0 < \sigma^2 < +\infty.$$

Let us consider the random walk

$$S_0 = 0,\ S_i = \sum_{j=1}^{i} X_j,\ i \in \mathbf{N}.$$

Let T be the first hitting time of the semi-axis $(-\infty,\ 0]$ by this random walk:

$$T = \min\{i > 0 : S_i \leq 0\}.$$

We introduce for each $n \in \mathbf{N}$ the random process $Y_n = \{Y_n(t),\ t \in [0, 1]\}$, where

$$Y_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}},\ t \in [0, 1].$$

Let $W = \{W(t),\ t \geq 0\}$ be a standard Brownian motion and $W^{ex} = \{W^{ex}(t),\ t \in [0, 1]\}$ be a normalised Brownian excursion. Let us denote β and γ the nearest zeros of a Brownian trajectory to point 1 on the left and right, respectively. Set $\Delta = \gamma - \beta$. It is known (see, for example, [1, sect. 1]) that

$$\left\{\frac{|W(\beta + t\Delta)|}{\sqrt{\Delta}},\ t \in [0, 1]\right\} \stackrel{D}{=} W^{ex}, \tag{1}$$

where the symbol $\stackrel{D}{=}$ means equality in distribution.

Let the symbol $\stackrel{D}{\to}$ denote convergence in distribution of a sequence of random elements. The following functional limit theorem is established in paper [2]: if assumptions 1 and 2 are valid, then, as $n \to \infty$,

$$\{Y_n \mid T = n\} \stackrel{D}{\to} W^{ex} \tag{2}$$

(this means convergence in distribution in the space $D[0, 1]$ with the Skorokhod topology).

Let $\xi(k,\ n)$ denote the number of hits of the walk $\{S_i,\ 0 \leq i \leq n\}$ into the state $k \in \mathbf{Z}$:

$$\xi(k,\ n) = |\{i : 0 \leq i \leq n,\ S_i = k\}|.$$

We call the random variable $\xi(k, n)$ the local time of the random walk $\{S_i, i \geq 0\}$ at point k up to time n. Introduce for each $n \in \mathbf{N}$ the random process $Z_n = \{Z_n(u), u \in \mathbf{R}\}$, where

$$Z_n(u) = \frac{\sigma\xi\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor, n\right)}{\sqrt{n}}, \; u \in \mathbf{R}.$$

We are interested in the asymptotic behaviour of the process considered under the condition that $T = n$.

For $u \in \mathbf{R}$ and $t \geq 0$, the Brownian motion W has almost surely the local time $l(u, t)$ defined by the formula

$$l(u, t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I_{[u-\varepsilon, u+\varepsilon]}(W(s))ds,$$

where $I_{[u-\varepsilon, u+\varepsilon]}(\cdot)$ is the indicator of the set $[u-\varepsilon, u+\varepsilon]$. Furthermore, it is possible to choose variants of $l(u, t)$ for different t and u such that the function $l(u, t)$ will be almost surely continuous with respect to $(u, t) \in \mathbf{R} \times [0, +\infty)$. It follows from this fact and relation (1) that, firstly, for each $u \geq 0$ and $t \in [0, 1]$, there exists almost surely the local time of the Brownian excursion W^{ex} defined by the formula

$$l^{ex}(u, t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{[u, u+\varepsilon]}(W^{ex}(s))ds,$$

and, secondly, it is possible to choose variants of $l^{ex}(u, t)$ for different t and u such that the function $l^{ex}(u, t)$ will be almost surely continuous with respect to $(u, t) \in [0, +\infty) \times [0, 1]$. This is assumed in what follows.

Theorem 1. *If assumptions 1 and 2 are valid, then, as* $n \to \infty$,

$$\{Z_n(u), u \geq 0 \mid T = n\} \xrightarrow{D} \{l^{ex}(u, 1), u \geq 0\}$$

(this means convergence in distribution in the space $D[0, v]$ *with the Skorokhod topology for arbitrary* $v > 0$*).*

As a corollary of theorem 1, the following results are obtained. Set for $x \in \mathbf{R}$

$$n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

For each $u > 0$, let us introduce the distribution function

$$T_u^{ex}(x) = 1 - 2\sqrt{2\pi} \sum_{l=1}^{\infty} \sum_{j=0}^{l-1} C_{l-1}^j \frac{x^j}{j!} n^{(j+2)}(2lu + x), \; x \geq 0$$

($T_u^{ex}(x) = 0$ for $x < 0$). We also introduce the distribution function

$$F^{ex}(x) = 1 + 2\sum_{l=1}^{\infty} e^{-2l^2x^2}\left(1 - 4l^2x^2\right), \; x > 0$$

($F^{ex}(x) = 0$ for $x \leq 0$). Note that the function $T_u^{ex}(x)$ is continuous for $x > 0$ and $T_u^{ex}(0) = F^{ex}(u)$.

Theorem 2. *If assumptions 1 and 2 are valid, then for every* $u \geq 0$ *and for all* $x \geq 0$

$$\lim_{n \to \infty} \mathbf{P}\left(Z_n(u) \leq x \mid T = n\right) = T_u^{ex}(x). \tag{3}$$

In particular,

$$\lim_{n \to \infty} \mathbf{P}\left(Z_n(u) = 0 \mid T = n\right) = F^{ex}(u). \tag{4}$$

Set

$$M_n = \sup_{u \geq 0} Z_n(u).$$

Theorem 3. *If assumptions 1 and 2 are valid, then for each* $x \geq 0$

$$\lim_{n \to \infty} \mathbf{P}\left(M_n \leq x \mid T = n\right) = F^{ex}\left(\frac{x}{2}\right).$$

This paper continues the author's series of papers, in which limit theorems on convergence to the local time of a Brownian meander [3], a stopped Brownian meander [4], a Brownian high jump [5], a Brownian bridge and a reflected Brownian bridge [6] are established. Finally, we note that if the random walk $\{S_i,\ i \geq 0\}$ is symmetric simple, convergence of one-dimensional distributions in theorem 1 was established in paper [7].

Auxiliary results

It is known (see, for example, [8, chap. 12, § 7 and chap. 18, § 5]) that if assumption 2 is valid, then $0 < \mathbf{E}|S_T| < +\infty$ and, as $n \to \infty$,

$$\mathbf{P}(T > n) \sim \frac{c_1}{\sqrt{n}}, \tag{5}$$

where

$$c_1 = \frac{\sqrt{2}\mathbf{E}|S_T|}{\sqrt{\pi}\sigma}.$$

It was shown in paper [9] that if assumptions 1 and 2 are valid, then, as $n \to \infty$,

$$\mathbf{P}(T = n) \sim \frac{c_1}{2n^{\frac{3}{2}}}. \tag{6}$$

Let for $x < 0$

$$T(x) = \min\{n : S_n \in (-\infty,\ x]\}.$$

Let us introduce the distribution function G: for $y > 0$

$$G(y) = \frac{1}{\mathbf{E}|S_T|}\Big(\mathbf{E}|S_T| - \mathbf{E}\big(|S_T| - y;\ |S_T| \geq y\big)\Big).$$

The following result is established in paper [2, lemma 2]: if assumptions 1 and 2 are valid, then for all $x, y > 0$, as $n \to \infty$,

$$P\Big(T\big(-\sigma\sqrt{n}x\big) = n,\ S_n + \sigma\sqrt{n}x > -y\Big) = \frac{1}{n\sqrt{2\pi}} x e^{-\frac{x^2}{2}} G(y)\big(1 + o(1)\big), \tag{7}$$

moreover, for a fixed $y > 0$, relation (7) holds uniformly over $x \geq a$, where a is an arbitrary positive number.

Recall that $W = \{W(t),\ t \geq 0\}$ is the Brownian motion and β is the nearest zero of a Brownian trajectory to point 1 on the left, set $\Delta_1 = 1 - \beta$. Let $W^{me} = \{W^{me}(t),\ t \in [0, 1]\}$ be a Brownian meander. It is known (see, for example, [1, sect. 1]) that

$$\left\{\frac{|W(\beta + t\Delta_1)|}{\sqrt{\Delta_1}},\ t \in [0, 1]\right\} \overset{D}{=} W^{me}. \tag{8}$$

It follows from relation (8) that, firstly, for $u \geq 0$ and $t \in [0, 1]$, there is almost surely the local time of the Brownian meander W^{me} defined by the formula

$$l^{me}(u, t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{[u,\ u+\varepsilon]}\big(W^{me}(s)\big) ds,$$

and, secondly, it is possible to choose such variants of $l^{me}(u, t)$ for different u and t that the function $l^{me}(u, t)$ will be almost surely continuous with respect to $(u, t) \in [0, +\infty) \times [0, 1]$. This is assumed in what follows.

It is also known (see, for example, [1, sect. 4]) that, as $\delta \downarrow 0$,

$$\{W^{me}(t),\ t \in [0, 1] \mid W^{me}(1) \leq \delta\} \xrightarrow{D} \{W^{ex}(t),\ t \in [0, 1]\} \tag{9}$$

(convergence in distribution in the space $C[0, 1]$ with uniform convergence topology).

Recall Iglehart functional limit theorem (see [10; 11]): if assumption 2 is valid, then, as $n \to \infty$,

$$\{Y_n(t),\ t \in [0, 1] \mid T > n\} \xrightarrow{D} \{W^{me}(t),\ t \in [0, 1]\}. \tag{10}$$

It is established in paper [3] that if assumptions 1 and 2 are valid, then, as $n \to \infty$,

$$\{Z_n(u),\ u \geq 0 \mid T > n\} \xrightarrow{D} \{l^{me}(u, 1),\ u \geq 0\}$$

(convergence in distribution in the space $D[0, v]$ with the Skorokhod topology for arbitrary $v > 0$). This relation can be generalised: if assumptions 1 and 2 are valid, then for an arbitrary $t \in (0, 1)$, as $n \to \infty$,

$$\left\{ Z_n(u, t),\ u \geq 0 \mid T > n \right\} \xrightarrow{D} \left\{ l^{me}(u, t),\ u \geq 0 \right\}, \tag{11}$$

where for $u \geq 0$

$$Z_n(u, t) = \frac{\sigma\xi\left(\lfloor u\sigma\sqrt{n} \rfloor, \lfloor nt \rfloor\right)}{\sqrt{n}}.$$

Relation (11) is also valid if the variable u (in left part of relation (11)) is replaced by $u + \alpha_n(u)$, where $\alpha_n(u)$ tends, as $n \to \infty$, to 0 uniformly over $u \geq 0$. Note that relations (10) and (11) can be combined into one: if assumptions 1 and 2 are valid, then for an arbitrary $t \in [0, 1]$, as $n \to \infty$,

$$\left\{ \left(Y_n(s),\ Z_n(u, t)\right),\ s \in [0,\ t],\ u \geq 0 \mid T > n \right\} \xrightarrow{D} \left\{ \left(W^{me}(s),\ l^{me}(u, t)\right),\ s \in [0,\ t],\ u \geq 0 \right\} \tag{12}$$

(convergence in distribution in the space $D[0,\ t] \times D[0,\ v]$ with the Skorokhod topology for arbitrary $v > 0$).

We introduce the following notation: N_s is a random variable having a normal distribution with parameters 0 and $s > 0$; $n_s(x)$ is distribution density of the random variable N_s, i. e.

$$n_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}},\ x \in \mathbf{R};$$

$N_s(a, b)$ is distribution of the random variable N_s over intervals, i. e.

$$N_s(a, b) = \mathbf{P}(a < N_s \leq b),\ a, b \in \mathbf{R},\ a \leq b$$

(herewith $N_0(0, b) = \frac{1}{2}$ for $b > 0$). We set for $s > 0$ and $x, y \in \mathbf{R}$

$$g_s(x, y) = n_s(x - y) - n_s(x + y),$$

$$h_s(x) = \frac{x}{s^{\frac{3}{2}}} \exp\left(-\frac{x^2}{2s}\right).$$

It is known (see, for example, [1, sect. 1]) that the Brownian meander W^{me} is an inhomogeneous Markov process with transition density $p^{me}(s, x; t, y)$, where for $0 < t \leq 1$ and $y > 0$

$$p^{me}(0, 0; t, y) = 2h_t(y)N_{1-t}(0, y), \tag{13}$$

and for $0 < s < t \leq 1$, $x > 0$ and $y > 0$

$$p^{me}(s, x; t, y) = g_{t-s}(x, y)\frac{N_{1-t}(0, y)}{N_{1-s}(0, x)}. \tag{14}$$

Set for $t \in [0, 1]$

$$W_t^{me} = \left\{ W^{me}(s),\ s \in [0,\ t] \right\},\ W_t^{ex} = \left\{ W^{ex}(s),\ s \in [0,\ t] \right\}$$

and for $t \in (0, 1)$

$$q_t(x) = \frac{h_t(x)}{\sqrt{2\pi} N_t(0, x)}.$$

Lemma 1. *Let $t \in (0, 1)$ and f be a measurable non-negative and bounded mapping $D[0,\ t]$ with the Skorokhod topology into $\mathbf{R}$, and let this mapping be almost surely continuous on trajectories of the process W_t^{ex}. Then*

$$\mathbf{E} f\left(W_t^{ex}\right) = \int_0^{+\infty} q_{1-t}(x)\, d_x \mathbf{E}\left(f\left(W_t^{me}\right);\ W^{me}(t) \leq x\right).$$

P r o o f. It follows from relation (9) by the continuous mapping theorem (see [12, theorem 5.1]) that, as $\delta \downarrow 0$,

$$\left\{f\left(W^{me}\right) \;\middle|\; W^{me}(1) \le \delta\right\} \xrightarrow{D} f\left(W^{ex}\right).$$

From where, according to the dominated convergence theorem, we find that

$$\mathbf{E} f\left(W_t^{ex}\right) = \lim_{\delta \downarrow 0} \mathbf{E}\left(f\left(W_t^{me}\right) \;\middle|\; W^{me}(1) \le \delta\right). \tag{15}$$

Since the Brownian meander is an inhomogeneous Markov process, then

$$\mathbf{E}\left(f\left(W_t^{me}\right) \;\middle|\; W^{me}(1) \le \delta\right) = \int_0^{+\infty} \frac{\mathbf{P}^{(t,\,x)}\left(W^{me}(1) \le \delta\right)}{\mathbf{P}\left(W^{me}(1) \le \delta\right)} d_x \mathbf{E}\left(f\left(W_t^{me}\right);\, W^{me}(t) \le x\right) \tag{16}$$

(the superscript of the symbol $\mathbf{P}^{(t,\,x)}$ means that the Brownian meander starts from point x at time t). Note that for $x > 0$

$$\mathbf{P}^{(t,\,x)}\left(W^{me}(1) \le \delta\right) = \int_0^{\delta} p^{me}(t,\, x;\, 1,\, y)\, dy, \tag{17}$$

$$\mathbf{P}\left(W^{me}(1) \le \delta\right) = \int_0^{\delta} p^{me}(0,\, 0;\, 1,\, y)\, dy. \tag{18}$$

Since, as $y \to 0$,

$$g_{1-t}(x,\, y) \sim \frac{2h_{1-t}(x)}{\sqrt{2\pi}} y,\ h_1(y) \sim y,$$

then (see relations (13) and (14))

$$p^{me}(t,\, x;\, 1,\, y) \sim \frac{h_{1-t}(x)\, y}{\sqrt{2\pi} N_{1-t}(0,\, x)},\ p^{me}(0,\, 0;\, 1,\, y) \sim y. \tag{19}$$

We find from formula (19) that, as $\delta \downarrow 0$,

$$\int_0^{\delta} p^{me}(t,\, x;\, 1,\, y)\, dy \sim \frac{h_{1-t}(x)\, \delta^2}{2\sqrt{2\pi} N_{1-t}(0,\, x)}, \tag{20}$$

$$\int_0^{\delta} p^{me}(0,\, 0;\, 1,\, y)\, dy \sim \frac{\delta^2}{2}. \tag{21}$$

It follows from relations (17), (18), (20) and (21) that for $x > 0$

$$\lim_{\delta \downarrow 0} \frac{\mathbf{P}^{(t,\,x)}\left(W^{me}(1) \le \delta\right)}{\mathbf{P}\left(W^{me}(1) \le \delta\right)} = q_{1-t}(x). \tag{22}$$

We show that for all $x > 0$ and $y \in (0,\, 1]$

$$\frac{p^{me}(t,\, x;\, 1,\, y)}{y} \le K(t), \tag{23}$$

where $K(t) > 0$ is independent of x, y. We find from relation (14) that

$$p^{me}(t,\, x;\, 1,\, y) = \frac{g_{1-t}(x,\, y)}{2N_{1-t}(0,\, x)} = \frac{n_{1-t}(y-x) - n_{1-t}(y+x)}{2\int_0^{x} n_{1-t}(u)\, du}. \tag{24}$$

Since $1 - e^{-a} \le a$ for $a \ge 0$, then

$$n_{1-t}(y-x) - n_{1-t}(y+x) = n_{1-t}(y-x)\left(1 - e^{\frac{-2xy}{1-t}}\right) \le \frac{2xy}{1-t} n_{1-t}(y-x). \tag{25}$$

It follows from relations (24) and (25) that

$$\frac{p^{me}(t,\, x;\, 1,\, y)}{y} \le \frac{x n_{1-t}(y-x)}{(1-t)\int\limits_0^x n_{1-t}(u)\,du}. \qquad (26)$$

Set $x_0 = \sqrt{2(1-t)\ln 2}$. It is not difficult to see that for $0 \le u \le x_0$

$$n_{1-t}(u) \ge \frac{1}{2\sqrt{2\pi(1-t)}}.$$

Therefore, if $0 < x \le x_0$, then

$$\int\limits_0^x n_{1-t}(u)\,du \ge \frac{x}{2\sqrt{2\pi(1-t)}}. \qquad (27)$$

We get from relations (25)–(27) that for $0 < x \le x_0$ and $y > 0$

$$\frac{p^{me}(t,\, x;\, 1,\, y)}{y} \le \frac{2}{1-t}\exp\left(-\frac{(y-x)^2}{2(1-t)}\right) \le \frac{2}{1-t},$$

i. e. relation (23) is valid for $0 < x \le x_0$ and $y > 0$. In view of inequality (26), for $x \ge x_0$ and $y > 0$

$$\frac{p^{me}(t,\, x;\, 1,\, y)}{y} \le \frac{x n_{1-t}(y-x)}{(1-t)\int\limits_0^{x_0} n_{1-t}(u)\,du}. \qquad (28)$$

It is easy to see that the numerator of the right-hand side of inequality (28) is bounded for $x > 0$ and $y \in (0,\, 1]$ by a positive constant depending on t. Hence, relation (23) is valid for $x \ge x_0$ and $y \in (0,\, 1]$. So, relation (23) is proved.

We have in view of relations (17) and (23) that for $\delta \in (0,\, 1]$ and any $x > 0$

$$\mathbf{P}^{(t,\, x)}\left(W^{me}(1) \le \delta\right) \le K(t)\frac{\delta^2}{2}. \qquad (29)$$

From relation (29), taking into account formula (21), we find that there exists such $\delta_0 > 0$ that for any $\delta \in (0,\, \delta_0)$ and any $x > 0$

$$\frac{\mathbf{P}^{(t,\, x)}\left(W^{me}(1) \le \delta\right)}{\mathbf{P}\left(W^{me}(1) \le \delta\right)} \le 2K(t). \qquad (30)$$

According to the dominated convergence theorem and due to relations (22) and (30)

$$\lim_{\delta \downarrow 0}\int\limits_0^{+\infty} \frac{\mathbf{P}^{(t,\, x)}\left(W^{me}(1) \le \delta\right)}{\mathbf{P}\left(W^{me}(1) \le \delta\right)} d_x\mathbf{E}\left(f\left(W_t^{me}\right);\, W^{me}(t) \le x\right) = \int\limits_0^{+\infty} q_{1-t}(x)\, d_x\mathbf{E}\left(f\left(W_t^{me}\right);\, W^{me}(t) \le x\right).$$

Therefore (see relation (16)),

$$\lim_{\delta \downarrow 0}\mathbf{E}\left(f\left(W_t^{me}\right) \;\middle|\; W^{me}(1) \le \delta\right) = \int\limits_0^{+\infty} q_{1-t}(x)\, d_x\mathbf{E}\left(f\left(W_t^{me}\right);\, W^{me}(t) \le x\right).$$

From where, remembering formula (15), we come to the statement of lemma 1.

Let $0 \le u < v$. Set for $t \in (0,\, 1]$ and $x \in D[0,\, t]$

$$f(x) = \int\limits_0^t I_{[u,\, v]}\left(x(s)\right) ds.$$

This functional is measurable, and continuous for almost all trajectories of the process W^{me} (see [12, app. II]). Set

$$\mu^{me}(u,\, v,\, t) = f\left(W_t^{me}\right),\ \mu^{ex}(u,\, v,\, t) = f\left(W_t^{ex}\right).$$

Note that $\mu^{me}(u, v, t)$ is the occupation time of the interval $[u, v]$ by the process W_t^{me}, and $\mu^{ex}(u, v, t)$ is the occupation time of the interval $[u, v]$ by the process W_t^{ex}. Moreover,

$$l^{me}(u, t) = \lim_{\varepsilon \downarrow 0} \frac{\mu^{me}(u, u+\varepsilon, t)}{\varepsilon}, \; l^{ex}(u, t) = \lim_{\varepsilon \downarrow 0} \frac{\mu^{ex}(u, u+\varepsilon, t)}{\varepsilon}.$$

Let $m \in N$, $\lambda_1, \ldots, \lambda_m \geq 0$ and $u_1, \ldots, u_m > 0$. Set

$$\Sigma_m^{me}(t) = \sum_{k=1}^{m} \lambda_k l^{me}(u_k, t),$$

$$\Sigma_m^{ex}(t) = \sum_{k=1}^{m} \lambda_k l^{ex}(u_k, t).$$

Lemma 2. *For arbitrary $t \in (0, 1)$*

$$\mathbf{E} e^{-\Sigma_m^{ex}(t)} = \int_0^{+\infty} q_{1-t}(x) d_x \mathbf{E}\left(e^{-\Sigma_m^{me}(t)}; W^{me}(t) \leq x\right).$$

P r o o f. By virtue of lemma 1 for $\varepsilon > 0$

$$\mathbf{E}\exp\left(-\sum_{k=1}^{m} \lambda_k \frac{\mu^{ex}(u_k, u_k+\varepsilon, t)}{\varepsilon}\right) = \int_0^{+\infty} q_{1-t}(x) d_x \mathbf{E}\left(\exp\left(-\sum_{k=1}^{m} \lambda_k \frac{\mu^{me}(u_k, u_k+\varepsilon)}{\varepsilon}\right); W^{me}(t) \leq x\right).$$

After integration in parts on the right side and passing to the limit, as $\varepsilon \to 0$, we obtain, according to the dominated convergence theorem, the statement of lemma 2.

For arbitrary $t \in (0, 1)$, we set

$$l_t^{me} = \left\{ l^{me}(u, t), \, u \geq 0 \right\}, \; l_t^{ex} = \left\{ l^{ex}(u, t), \, u \geq 0 \right\}.$$

The following statement is of independent interest.

Corollary 1. *For arbitrary $t \in (0, 1)$, $v > 0$ and any Borel set A of the space $D[0, v]$*

$$\mathbf{P}\left(l_t^{ex} \in A\right) = \int_0^{+\infty} q_{1-t}(x) d_x \mathbf{P}\left(l_t^{me} \in A, \, W^{me}(t) \leq x\right). \tag{31}$$

P r o o f. By virtue of lemma 2, relation (31) is valid for cylindrical sets A of the space $D[0, v]$. Since both the left and right sides of relation (31) are probability measures, the required statement follows from Caratheodory theorem.

Proof of theorem 1

We establish in the beginning convergence of finite-dimensional distributions. Let us fix $m \in \mathbf{N}$ and $u_1, \ldots, u_m > 0$. We show that, as $n \to \infty$,

$$\left(Z_n(u_1), \ldots, Z_n(u_m) \mid T = n\right) \xrightarrow{D} \left(l^{ex}(u_1), \ldots, l^{ex}(u_m)\right). \tag{32}$$

For an arbitrary $t \in (0, 1)$, we write the representation: for $u \geq 0$

$$Z_n(u) = Z_n(u, t) + \zeta_n(u, t), \tag{33}$$

where

$$\zeta_n(u, t) = \frac{\sigma\xi\left(\lfloor u\sigma\sqrt{n} \rfloor, \lfloor nt \rfloor + 1, n\right)}{\sqrt{n}},$$

and for $l \leq n$

$$\xi(j, l, n) = \left|\left\{ i \in \{l, \ldots, n\} : S_i = j \right\}\right|.$$

First, we show that

$$\lim_{n \to \infty} \mathbf{E}\left(e^{-\Sigma_m(n, t)} \mid T = n\right) = \mathbf{E} e^{-\Sigma_m^{ex}(t)}, \tag{34}$$

where

$$\Sigma_m(n,\ t) = \sum_{k=1}^{m} \lambda_k Z_n(u_k,\ t)$$

and $\lambda_1, \ldots, \lambda_m \geq 0$. For arbitrary $y, z > 0$

$$0 \leq \mathbf{E}\left(e^{-\Sigma_m(n,\,t)} \,\middle|\, T = n\right) - E(n,\ y,\ z) \leq E_1(n,\ z) + E_2(n,\ y), \tag{35}$$

where

$$E(n,\ y,\ z) = \mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ Y_n(t) > z,\ S_n > -y \,\middle|\, T = n\right),$$

$$E_1(n,\ z) = \mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ Y_n(t) \leq z \,\middle|\, T = n\right),$$

$$E_2(n,\ y) = \mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ S_n \leq -y \,\middle|\, T = n\right).$$

Note that

$$0 \leq E_1(n,\ z) \leq \mathbf{P}\left(Y_n(t) \leq z \,\middle|\, T = n\right). \tag{36}$$

Applying relation (2) to inequality (36), we find that

$$\limsup_{n \to \infty} E_1(n,\ z) \leq \mathbf{P}\left(W^{ex}(t) \leq z\right),$$

hence,

$$\lim_{z \to 0} \limsup_{n \to \infty} E_1(n,\ z) = 0. \tag{37}$$

It is shown in paper [2, relation (4.7)] that

$$\left\{S_n \,\middle|\, T = n\right\} \xrightarrow{D} S, \tag{38}$$

where S is some random variable, and $\mathbf{P}(S > -y) = G(y)$ for $y > 0$. Since

$$0 \leq E_2(n,\ y) \leq \mathbf{P}\left(S_n \leq -y \,\middle|\, T = n\right),$$

then (see formula (38))

$$\limsup_{n \to \infty} E_2(n,\ y) \leq \mathbf{P}(S \leq -y)$$

for any $y > 0$, with the exception of no more than a countable set. Hence,

$$\lim_{y \to \infty} \limsup_{n \to \infty} E_2(n,\ y) = 0. \tag{39}$$

Set $n(t) = n - \lfloor nt \rfloor$. By the Markov property of random walks

$$n^{\frac{3}{2}} \mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ Y_n(t) > z,\ S_n > -y,\ T = n\right) = -\int_{(z,\,+\infty)} f_n(x)\, d_x \mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ Y_n(t) > x \,\middle|\, T > \lfloor nt \rfloor\right), \tag{40}$$

where for $x > 0$

$$f_n(x) = n^{\frac{3}{2}} \mathbf{P}\left(T > \lfloor nt \rfloor\right) \mathbf{P}\left(T\left(-\sigma\sqrt{n}x\right) = n(t),\ S_{n(t)} + \sigma\sqrt{n}x > -y\right).$$

We have by virtue of relations (5) and (7) that for arbitrary $x, y > 0$, as $n \to \infty$,

$$f_n(x) = \frac{c_1 G(y)}{\sqrt{2\pi t}} h_{1-t}(x)\left(1 + o(1)\right), \tag{41}$$

moreover, for a fixed y, this relation is performed uniformly over $x > z$.

Set

$$\tilde{\Sigma}_m^{me}(t) = \sum_{k=1}^{m} \lambda_k \tilde{l}^{me}(u_k, t),$$

where for $u > 0$

$$\tilde{l}^{me}(u, t) = \sqrt{t} l^{me}\left(\frac{u}{\sqrt{t}}, 1\right).$$

In view of formula (12)

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) > x \,\middle|\, T > \lfloor nt \rfloor\right) = \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; \sqrt{t}W^{me}(1) > x\right). \tag{42}$$

It follows from relations (40)–(42) that

$$\lim_{n\to\infty} n^{\frac{3}{2}} \mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) > z, S_n > -y, T = n\right) = \frac{c_1 G(y)}{\sqrt{2\pi t}} \int_{(z, +\infty)} h_{1-t}(x) d_x \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(1) \le \frac{x}{\sqrt{t}}\right). \tag{43}$$

Due to formulas (6) and (43)

$$\lim_{n\to\infty} E(n, y, z) = G(y)\sqrt{\frac{2}{\pi t}} \int_{(z, +\infty)} h_{1-t}(x) d_x \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(1) \le \frac{x}{\sqrt{t}}\right).$$

Therefore,

$$\lim_{y\to\infty} \lim_{z\to 0} \lim_{n\to\infty} E(n, y, z) = \int_0^{+\infty} \sqrt{\frac{2}{\pi t}} h_{1-t}(x) d_x \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(1) \le \frac{x}{\sqrt{t}}\right). \tag{44}$$

We find from relations (35), (37), (39) and (44) that

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\Sigma_m(n, t)} \,\middle|\, T = n\right) = \sqrt{\frac{2}{\pi t}} \int_0^{+\infty} h_{1-t}(x) d_x \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(1) \le \frac{x}{\sqrt{t}}\right). \tag{45}$$

Lemma 3. *For arbitrary $t \in (0, 1)$ and any $x > 0$*

$$\mathbf{E}\left(e^{-\Sigma_m^{me}(t)}; W^{me}(t) \le x\right) = \frac{2}{\sqrt{t}} \int_0^{x} N_{1-t}(0, z) d_z \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(1) \le \frac{z}{\sqrt{t}}\right).$$

P r o o f. In view of formula (12)

$$\mathbf{E}\left(e^{-\Sigma_m^{me}(t)}; W^{me}(t) \le x\right) = \lim_{n\to\infty} \frac{\mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) \le x, T > n\right)}{\mathbf{P}(T > n)}. \tag{46}$$

Set

$$L_n = \min_{0 \le i \le n} S_i.$$

By the Markov property of random walks

$$\mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) \le x, T > n\right) = \int_0^{x} \mathbf{P}\left(\frac{L_{n(t)}}{\sigma\sqrt{n}} > -z\right) d_z \mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) \le z, T > \lfloor nt \rfloor\right). \tag{47}$$

By virtue of the Donsker – Prokhorov invariance principle

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{L_{n(t)}}{\sigma\sqrt{n}} > -z\right) = 2N_1\left(0, \frac{z}{\sqrt{1-t}}\right) = 2N_{1-t}(0, z), \tag{48}$$

and the first equality holds uniformly over $z \in (0, x]$. Due to formula (12)

$$\lim_{n\to\infty} \frac{\mathbf{E}\left(e^{-\Sigma_m(n, t)}; Y_n(t) \le z, T > \lfloor nt \rfloor\right)}{\mathbf{P}\left(T > \lfloor nt \rfloor\right)} = \mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)}; W^{me}(t) \le \frac{z}{\sqrt{t}}\right). \tag{49}$$

It follows from relations (5) and (47)–(49) that

$$\lim_{n\to\infty}\frac{\mathbf{E}\left(e^{-\Sigma_m(n,\,t)};\ Y_n(t)\le x,\ T>n\right)}{\mathbf{P}(T>n)}=\frac{2}{\sqrt{t}}\int_0^x N_{1-t}(0,\,z)\,d_z\mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)};\ W^{me}(t)\le\frac{z}{\sqrt{t}}\right). \tag{50}$$

Relations (46) and (50) imply the statement of lemma 3.

Let us continue the proof of theorem 1. We show that

$$\sqrt{\frac{2}{\pi t}}\int_0^{+\infty}h_{1-t}(x)\,d_x\mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)};\ W^{me}(1)\le\frac{x}{\sqrt{t}}\right)=\mathbf{E}e^{-\Sigma_m^{ex}(t)}. \tag{51}$$

By lemma 2

$$\int_0^{+\infty}q_{1-t}(x)\,d_x\mathbf{E}\left(e^{-\Sigma_m^{me}(t)};\ W^{me}(t)\le x\right)=\mathbf{E}e^{-\Sigma_m^{ex}(t)}. \tag{52}$$

By lemma 3

$$\int_0^{+\infty}q_{1-t}(x)\,d_x\mathbf{E}\left(e^{-\Sigma_m^{me}(t)};\ W^{me}(t)\le x\right)=\frac{2}{\sqrt{t}}\int_0^{+\infty}q_{1-t}(x)\,N_{1-t}(0,\,x)\,d_x\mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)};\ W^{me}(1)\le\frac{x}{\sqrt{t}}\right)=$$

$$=\sqrt{\frac{2}{\pi t}}\int_0^{+\infty}h_{1-t}(x)\,d_x\mathbf{E}\left(e^{-\tilde{\Sigma}_m^{me}(t)};\ W^{me}(1)\le\frac{x}{\sqrt{t}}\right). \tag{53}$$

We obtain from formulas (52) and (53) the required relation (51), and from formulas (45) and (51) we have the required relation (34).

By the continuity theorem for characteristic functions, we find from relation (34) that, as $n\to\infty$,

$$\left(Z_n(u_1,\,t),\,\ldots,\,Z_n(u_m,\,t)\,\middle|\,T=n\right)\xrightarrow{D}\left(l^{ex}(u_1,\,t),\,\ldots,\,l^{ex}(u_m,\,t)\right). \tag{54}$$

Since the function $l^{ex}(u,\,t)$ is continuous with respect t with probability 1, then almost surely, as $t\to1$,

$$\left(l^{ex}(u_1,\,t),\,\ldots,\,l^{ex}(u_m,\,t)\right)\to\left(l^{ex}(u_1,1),\ldots,l^{ex}(u_m,1)\right). \tag{55}$$

We show that for an arbitrary $u>0$

$$\lim_{t\to1}\limsup_{n\to\infty}\mathbf{P}\left(\zeta_n(u,\,t)>0\,\middle|\,T=n\right)=0. \tag{56}$$

If $\sup\limits_{t\le s\le1}Y_n(s)<u$, then $\zeta_n(u,\,t)=0$. Therefore,

$$\mathbf{P}\left(\zeta_n(u,\,t)>0\,\middle|\,T=n\right)=\mathbf{P}\left(\zeta_n(u,\,t)>0,\ \sup_{t\le s\le1}Y_n(s)\ge u\,\middle|\,T=n\right). \tag{57}$$

Further,

$$\mathbf{P}\left(\zeta_n(u,\,t)>0,\ \sup_{t\le s\le1}Y_n(s)\ge u\,\middle|\,T=n\right)\le\mathbf{P}\left(\sup_{t\le s\le1}Y_n(s)\ge u\,\middle|\,T=n\right). \tag{58}$$

By virtue of relation (2)

$$\lim_{n\to\infty}\mathbf{P}\left(\sup_{t\le s\le1}Y_n(s)\ge u\,\middle|\,T=n\right)=\mathbf{P}\left(\sup_{t\le s\le1}W^{ex}(s)\ge u\right). \tag{59}$$

It follows from formulas (57)–(59) that

$$\limsup_{n\to\infty}\mathbf{P}\left(\zeta_n(u,\,t)>0\,\middle|\,T=n\right)\le\mathbf{P}\left(\sup_{t\le s\le1}W^{ex}(s)\ge u\right). \tag{60}$$

Since almost surely $\sup\limits_{t\le s\le1}W^{ex}(s)\to0$, as $t\to1$, then

$$\lim_{t\to1}\mathbf{P}\left(\sup_{t\le s\le1}W^{ex}(s)\ge u\right)=0. \tag{61}$$

We obtain from formulas (60) and (61) the required relation (56).

In view of relation (56)

$$\lim_{t\to 1}\limsup_{n\to\infty}\mathbf{P}\left(\max_{1\le k\le m}\zeta_n\left(u_k,\, t\right)>0\,\middle|\, T=n\right)=0. \tag{62}$$

We find from formulas (33) and (62) that

$$\lim_{t\to 1}\limsup_{n\to\infty}\mathbf{P}\left(\max_{1\le k\le m}\left|Z_n\left(u_k\right)-Z_n\left(u_k,\, t\right)\right|>0\,\middle|\, T=n\right)=0. \tag{63}$$

By applying formulas (54), (55), (63) and theorem 4.2 from paper [12], we obtain the required relation (32).

Let us fix $v>0$. We introduce a continuity module for $x\in D\left[0,\, v\right]$: for $\delta>0$

$$w_x\left(\delta\right)=\sup_{u_1,\, u_2\in\left[0,\, v\right]:\, \left|u_1-u_2\right|\le\delta}\left|x\left(u_1\right)-x\left(u_2\right)\right|.$$

We show that for an arbitrary $\varepsilon>0$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(w_{Z_n}\left(\delta\right)\ge\varepsilon\,\middle|\, T=n\right)=0. \tag{64}$$

Note (see formula (33)) that for $t\in\left(0,\, 1\right)$

$$w_{Z_n}\left(\delta\right)\le w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)+w_{\zeta_n\left(\cdot,\, t\right)}\left(\delta\right). \tag{65}$$

Due to relation (65)

$$\mathbf{P}\left(w_{Z_n}\left(\delta\right)\ge\varepsilon\,\middle|\, T=n\right)\le\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2}\,\middle|\, T=n\right)+\mathbf{P}\left(w_{\zeta_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2}\,\middle|\, T=n\right). \tag{66}$$

First, we show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2}\,\middle|\, T=n\right)=0. \tag{67}$$

Note that for arbitrary $y,\, z>0$

$$0\le\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2}\,\middle|\, T=n\right)-P\left(n,\, y,\, z\right)\le P_1\left(n,\, z\right)+P_2\left(n,\, y\right), \tag{68}$$

where

$$P\left(n,\, y,\, z\right)=\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2},\, Y_n\left(t\right)>z,\, S_n>-y\,\middle|\, T=n\right),$$

$$P_1\left(n,\, z\right)=\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2},\, Y_n\left(t\right)\le z\,\middle|\, T=n\right),$$

$$P_2\left(n,\, y\right)=\mathbf{P}\left(w_{Z_n\left(\cdot,\, t\right)}\left(\delta\right)\ge\frac{\varepsilon}{2},\, S_n\le -y\,\middle|\, T=n\right).$$

Since

$$P_1\left(n,\, z\right)\le\mathbf{P}\left(Y_n\left(t\right)\le z\,\middle|\, T=n\right),$$

then (see relation (2))

$$\limsup_{n\to\infty}P_1\left(n,\, z\right)\le\mathbf{P}\left(W^{ex}\left(t\right)\le z\right).$$

Therefore,

$$\lim_{z\to 0}\limsup_{n\to\infty}P_1\left(n,\, z\right)=0. \tag{69}$$

Since

$$P_2\left(n,\, y\right)\le\mathbf{P}\left(S_n\le -y\,\middle|\, T=n\right),$$

then (see formula (38))

$$\limsup_{n\to\infty} P_2(n, y) \le \mathbf{P}(S \le -y)$$

for any $y > 0$, with the exception of no more than a countable set. Therefore,

$$\lim_{y\to\infty} \limsup_{n\to\infty} P_2(n, y) = 0. \tag{70}$$

By the Markov property of random walks

$$n^{\frac{3}{2}}\mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2},\ Y_n(t) > z,\ S_n > -y,\ T = n\right) = -\int\limits_{(z,\, +\infty)} f_n(x)\, d_x \mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2},\ Y_n(t) > x \,\middle|\, T > \lfloor nt \rfloor\right), \tag{71}$$

where $f_n(x)$ is the same function as in relation (40). We have in view of formula (41) that for a fixed $y > 0$, all $x > z$ and all sufficiently large n

$$f_n(x) \le \frac{2c_1 G(y)}{\sqrt{2\pi t}} h_{1-t}(x) \le K(t), \tag{72}$$

where $K(t) > 0$ is independent of n, x. We find from formulas (71) and (72) that for a fixed $y > 0$ and all sufficiently large n

$$n^{\frac{3}{2}}\mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2},\ Y_n(t) > z,\ S_n > -y,\ T = n\right) \le$$

$$\le K(t)\,\mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2},\ Y_n(t) > z \,\middle|\, T > \lfloor nt \rfloor\right) \le K(t)\,\mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2} \,\middle|\, T > \lfloor nt \rfloor\right). \tag{73}$$

It follows from relation (11) that

$$\lim_{\delta\to 0} \lim_{n\to\infty} \mathbf{P}\left(w_{Z_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2} \,\middle|\, T > \lfloor nt \rfloor\right) = 0. \tag{74}$$

We get from formulas (73) and (74) that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} n^{\frac{3}{2}}\mathbf{P}\left(w_{Z_n^{(t)}}(\delta) \ge \frac{\varepsilon}{2},\ Y_n(t) > z,\ S_n > -y,\ T = n\right) = 0$$

and, hence (see relation (6)),

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P(n, y, z) = 0. \tag{75}$$

We obtain from formulas (68)–(70) and (75) the required relation (67).

Now let us show that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbf{P}\left(w_{\zeta_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2} \,\middle|\, T = n\right). \tag{76}$$

Note that for an arbitrary $L > 0$

$$0 \le \mathbf{P}\left(w_{\zeta_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2} \,\middle|\, T = n\right) - \sum_{l=0}^{\lfloor L \rfloor} q(n, l) \le Q(n, L), \tag{77}$$

where

$$Q(n, L) = \mathbf{P}(S_n < -L \mid T = n)$$

and for $l \in \{0, 1, 2, \ldots\}$

$$q(n, l) = \mathbf{P}\left(w_{\zeta_n(\cdot,\, t)}(\delta) \ge \frac{\varepsilon}{2},\ S_n = -l \,\middle|\, T = n\right).$$

By virtue of formula (38)

$$\lim_{n\to\infty} Q(n, L) = \mathbf{P}(S < -L)$$

for any L, with the exception of no more than a countable set, hence,

$$\lim_{L\to\infty}\lim_{n\to\infty} Q(n, L) = 0. \tag{78}$$

We show that for a fixed $l \in \{0, 1, 2, \ldots\}$

$$\lim_{\delta\to 0}\limsup_{n\to\infty} q(n, l) = 0. \tag{79}$$

Set

$$\tilde{S}_0 = 0,\ \tilde{S}_1 = S_{n-1} - S_n,\ \tilde{S}_2 = S_{n-2} - S_n,\ \ldots,\ \tilde{S}_n = S_0 - S_n = -S_n.$$

Note that

$$\left(\tilde{S}_1, \ldots, \tilde{S}_n\right) \overset{D}{=} -(S_1, \ldots, S_n).$$

Let $\tilde{\xi}(k, m)$, where $m \in \{0, 1, \ldots, n\}$, be the number of hits of the random walk $\left\{\tilde{S}_i, 0 \le i \le m\right\}$ in the state $k \in \mathbf{Z}$, i. e.

$$\tilde{\xi}(k, m) = \left|\left\{i : 0 \le i \le m,\ \tilde{S}_i = k\right\}\right|.$$

If $S_n = -l$, then for $m \in \{0, 1, \ldots, n\}$

$$\tilde{\xi}(k, m) = \left|\left\{i : 0 \le i \le m,\ S_{n-i} - S_n = k\right\}\right| = \left|\left\{i : 0 \le i \le m,\ S_{n-i} = k - l\right\}\right| =$$

$$= \left|\left\{i : n - m \le i \le n,\ S_i = k - l\right\}\right| = \xi(k - l, n - m, n). \tag{80}$$

Let $\tilde{T}$ mean the same for the walk $\left\{\tilde{S}_i\right\}$ as T means for the walk $\{S_i\}$. Then

$$\{S_n = -l, T = n\} = \{S_1 > 0, \ldots, S_{n-1} > 0,\ S_n = -l\} = \left\{\tilde{S}_{n-1} - \tilde{S}_n > 0, \ldots, \tilde{S}_1 - \tilde{S}_n > 0,\ \tilde{S}_n = l\right\} =$$

$$= \left\{\tilde{S}_1 > l, \ldots, \tilde{S}_{n-1} > l,\ \tilde{S}_n = l\right\} \subset \left\{\tilde{S}_1 > 0, \ldots, \tilde{S}_{n-1} > 0,\ \tilde{S}_n = l\right\} = \left\{\tilde{S}_n = l,\ \tilde{T} > n\right\}. \tag{81}$$

Set for $u \in \mathbf{R}$

$$\tilde{Z}_n(u, t) = \frac{\sigma\tilde{\xi}\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor + l, \lfloor nt \rfloor\right)}{\sqrt{n}}.$$

If $S_n = -l$, then by virtue of relation (80)

$$\zeta_n(u, t) = \frac{\sigma\tilde{\xi}\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor + l, n - \lfloor nt \rfloor - 1\right)}{\sqrt{n}} = \frac{\sigma\tilde{\xi}\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor + l, \lfloor n(1 - t_n)\rfloor\right)}{\sqrt{n}} = \tilde{Z}_n(u, 1 - t_n), \tag{82}$$

where t_n is found from the equality $n - \lfloor nt \rfloor - 1 = \lfloor n(1 - t_n)\rfloor$. Since $|t_n - t| \le \dfrac{1}{n}$, then

$$\left|\tilde{\xi}\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor + l, \lfloor n(1 - t_n)\rfloor\right) - \tilde{\xi}\left(\left\lfloor u\sigma\sqrt{n}\right\rfloor + l, \lfloor n(1 - t)\rfloor\right)\right| \le 1$$

and, hence,

$$\left|\tilde{Z}_n(u, 1 - t_n) - \tilde{Z}_n(u, 1 - t)\right| \le \frac{\sigma}{\sqrt{n}}. \tag{83}$$

It follows from inequality (83) that

$$\left|w_{\tilde{Z}_n(\cdot,\, 1 - t_n)}(\delta) - w_{\tilde{Z}_n(\cdot,\, 1 - t)}(\delta)\right| \le \frac{2\sigma}{\sqrt{n}}. \tag{84}$$

If $S_n = -l$, then by virtue of relations (82) and (84)

$$\left|w_{\zeta_n(\cdot,\, t)}(\delta) - w_{\tilde{Z}_n(\cdot,\, 1 - t)}(\delta)\right| \le \frac{2\sigma}{\sqrt{n}}. \tag{85}$$

To prove relation (79), it is enough to establish (see formula (6)) that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n^{\frac{3}{2}} \mathbf{P}\left(w_{\zeta_n(\cdot,\, t)}(\delta) \geq \frac{\varepsilon}{2},\ S_n = -l,\ T = n \right) = 0. \tag{86}$$

In turn, to prove relation (86), it is enough to establish (see formulas (81), (82) and (85)) that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n^{\frac{3}{2}} \mathbf{P}\left(w_{\tilde{Z}_n(u,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ \tilde{S}_n = l,\ \tilde{T} > n \right) = 0 \tag{87}$$

or, equivalently,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n^{\frac{3}{2}} \mathbf{P}\left(w_{Z_n(u,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ S_n = l,\ T > n \right) = 0. \tag{88}$$

It is not difficult to show (see the proof of relation (4.20) in paper [2]) that there are constants $K > 0$ and $L \in \mathbf{N}$ such that

$$\mathbf{P}\left(w_{Z_n(\cdot,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ S_n = l,\ T > n \right) \leq K \sum_{i=1}^{L} \mathbf{P}\left(w_{Z_n(\cdot,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ T = n + i \right). \tag{89}$$

It follows from formula (6) and the proved relation (67) that for each $i \in \{1, \ldots, l\}$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n^{\frac{3}{2}} \mathbf{P}\left(w_{Z_{n+i}(\cdot,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ T = n + i \right) = 0. \tag{90}$$

We show that for each $i \in \{1, \ldots, l\}$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n^{\frac{3}{2}} \mathbf{P}\left(w_{Z_n(\cdot,\, 1-t)}(\delta) \geq \frac{\varepsilon}{2},\ T = n + i \right) = 0. \tag{91}$$

Set

$$Z_{n+i}^{*}(u, 1-t) = \frac{\sigma \xi\left(\left\lfloor u\sigma\sqrt{n+i} \right\rfloor, \left\lfloor n(1-t) \right\rfloor \right)}{\sqrt{n+i}}.$$

Since $\lfloor (n+i)t \rfloor - \lfloor nt \rfloor \leq i + 1$, then

$$Z_{n+i}(u, 1-t) - Z_{n+i}^{*}(u, 1-t) \leq \sigma \frac{i+1}{\sqrt{n+i}}$$

and, hence,

$$\left| w_{Z_{n+i}(\cdot,\, 1-t)}(\delta) - w_{Z_{n+i}^{*}(\cdot,\, 1-t)}(\delta) \right| \leq \frac{2\sigma(i+1)}{\sqrt{n+i}}. \tag{92}$$

Further,

$$w_{Z_{n+i}^{*}(\cdot,\, 1-t)}(\delta) = \sqrt{\frac{n}{n+i}}\, w_{Z_n(\cdot,\, 1-t)}\left(\delta\sqrt{\frac{n+i}{n}} \right). \tag{93}$$

By virtue of formulas (92) and (93)

$$\left| w_{Z_{n+i}(\cdot,\, 1-t)}(\delta) - \sqrt{\frac{n}{n+i}}\, w_{Z_n(\cdot,\, 1-t)}\left(\delta\sqrt{\frac{n+i}{n}} \right) \right| \leq \frac{2\sigma(i+1)}{\sqrt{n+i}}$$

and, hence,

$$\left| \sqrt{\frac{n+i}{n}}\, w_{Z_{n+i}(\cdot,\, 1-t)}\left(\delta\sqrt{\frac{n}{n+i}} \right) - w_{Z_n(\cdot,\, 1-t)}(\delta) \right| \leq \frac{2\sigma(i+1)}{\sqrt{n}}.$$

Thus,

$$w_{Z_n(\cdot,\, 1-t)}(\delta) \leq \sqrt{\frac{n+i}{n}}\, w_{Z_{n+i}(\cdot,\, 1-t)}\left(\delta\sqrt{\frac{n}{n+i}} \right) + \frac{2\sigma(i+1)}{\sqrt{n}}$$

and, hence, for sufficiently large n

$$w_{Z_n(\cdot,\,1-t)}(\delta) \le 2w_{Z_{n+i}(\cdot,\,1-t)}(\delta) + \frac{2\sigma(i+1)}{\sqrt{n}}. \tag{94}$$

It follows from formulas (90) and (94) the required relation (91).

From formulas (89) and (91) we have relation (88) and, hence, relations (87) and (86). In turn, formula (86) implies the required relation (79). Further, from formulas (77)–(79) we find the required relation (76). Finally, from formulas (66), (67) and (76) we find the required relation (64). From relations (32) and (64) according to theorem 15.5 from book [12] we obtain the statement of theorem 1.

Proof of theorem 2

By theorem 1, as $n \to \infty$,

$$\{Z_n(u) \mid T = n\} \xrightarrow{D} l^{ex}(u, 1). \tag{95}$$

Set for $x \in \mathbf{R}$

$$F_u(x) = \mathbf{P}\left(l^{ex}(u, 1) \le x\right).$$

It follows from relation (95) that

$$\lim_{n\to\infty} \mathbf{P}\left(Z_n(u) \le x \mid T = n\right) = F_u(x) \tag{96}$$

for all x, which are points of continuity of the function $F_u(\cdot)$. It is proved in paper [7, relations (170) and (171)] that for any $u > 0$ and for all $x \ge 0$

$$F_u(x) = T_u^{ex}(x). \tag{97}$$

The function $T_u^{ex}(x)$ is continuous for all $x > 0$ and, hence, relation (96) is valid for all $x > 0$. Formulas (96) and (97) imply relation (3) for $x > 0$.

Let us prove the relation (4). We have due to formula (33) that for $t \in (0, 1)$

$$\left|\mathbf{P}\left(Z_n(u) = 0 \mid T = n\right) - P_1(n, t, u)\right| \le 1 - P_2(n, t, u), \tag{98}$$

where

$$P_1(n, t, u) = \mathbf{P}\left(Z_n(u, t) = 0 \mid T = n\right),$$

$$P_2(n, t, u) = \mathbf{P}\left(\zeta_n(u, t) = 0 \mid T = n\right).$$

Similarly to relation (34), it can be shown that

$$\lim_{n\to\infty} P_1(n, t, u) = \mathbf{P}\left(l_t^{ex}(u) = 0\right). \tag{99}$$

Note that

$$\mathbf{P}\left(l_t^{ex}(u) = 0\right) = \mathbf{P}\left(\sup_{0\le s\le t} W^{ex}(s) \le u\right). \tag{100}$$

Trajectories of the Brownian excursion W^{ex} are almost surely continuous, and therefore,

$$\lim_{t\to 1} \mathbf{P}\left(\sup_{0\le s\le t} W^{ex}(s) \le u\right) = \mathbf{P}\left(\sup_{0\le s\le 1} W^{ex}(s) \le u\right). \tag{101}$$

It is known (see [1, sect. 6]) that

$$\mathbf{P}\left(\sup_{0\le s\le 1} W^{ex}(s) \le u\right) = F^{ex}(u). \tag{102}$$

It follows from formulas (100)–(102) that

$$\lim_{t\to 1} \mathbf{P}\left(l_t^{ex}(u) = 0\right) = F^{ex}(u). \tag{103}$$

Applying relation (103) to relation (99), we find that

$$\lim_{t\to 1} \lim_{n\to\infty} P_1(n, t, u) = F^{ex}(u). \tag{104}$$

It follows from relation (56) that

$$\lim_{t\to 1}\liminf_{n\to\infty} P_2(n,\ t,\ u) = 1. \tag{105}$$

From formulas (98), (104) and (105) we obtain the required relation (4). Theorem 2 is proved.

Proof of theorem 3

By virtue of theorem 1, as $n\to\infty$,

$$\sup_{u\ge 0} Z_n(u) \xrightarrow{D} \sup_{u\ge 0} l^{ex}(u,\ 1).$$

Hence,

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{u\ge 0} Z_n(u) \le x\right) = \mathbf{P}\left(\sup_{u\ge 0} l^{ex}(u,\ 1) \le x\right) \tag{106}$$

for all $x \ge 0$, which are points of continuity of the right-hand side. It is known (see, for example, [13, sect. 4.2]) that

$$\sup_{u\ge 0} l^{ex}(u,\ 1) \stackrel{D}{=} 2 \sup_{s\in[0,\ 1]} W^{ex}(s). \tag{107}$$

We find from formulas (102) and (107) that for $x \ge 0$

$$\mathbf{P}\left(\sup_{u\ge 0} l^{ex}(u,\ 1) \le x\right) = \mathbf{P}\left(\sup_{s\in[0,\ 1]} W^{ex}(s) \le \frac{x}{2}\right) = F^{ex}\left(\frac{x}{2}\right). \tag{108}$$

Since the right-hand side of relation (108) is continuous, then relation (106) holds for all $x \ge 0$. We obtain from relations (106) and (108) the statement of theorem 3.

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