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КВАНТОВЫЕ ФАЗЫ ДЛЯ ЭЛЕКТРИЧЕСКИХ ЗАРЯДОВ И ЭЛЕКТРИЧЕСКИХ (МАГНИТНЫХ) ДИПОЛЕЙ: ФИЗИЧЕСКИЙ СМЫСЛ И ПРИМЕНЕНИЕ

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Рассматривается физический смысл квантовых фазовых эффектов для точечных зарядов и электрических или магнитных диполей в электромагнитном поле. В настоящее время известно восемь таких эффектов: четыре из них (магнитный и электрический эффекты Ааронова – Бома для электронов, эффект Ааронова – Кашера для движущегося магнитного диполя и эффект Хе – Маккеллара – Уилкенса для движущегося электрического диполя) были открыты в XX в., а четыре новых квантовых фазовых эффекта недавно обнаружены нашей группой (А. Л. Холмецкий, О. В. Мисевич, Т. Ярман). При анализе их физического смысла мы принимаем, что квантовая фаза для диполя представляет собой суперпозицию квантовых фаз для каждого заряда, составляющих диполь. При таком подходе мы демонстрируем неспособность уравнения Шрёдингера для заряженной частицы в электромагнитном поле описывать новые квантовые фазовые эффекты, когда используется стандартное определение оператора импульса. Далее мы показываем, что согласованное описание квантовых фазовых эффектов для движущихся частиц достигается при соответствующем переопределении этого оператора, когда канонический импульс частицы в электромагнитном поле заменяется импульсом электромагнитного поля. Обсуждаются некоторые следствия полученных результатов.

Ключевые слова: квантовые фазовые эффекты; электрический диполь; магнитный диполь; уравнение Шрёдингера; оператор импульса.

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QUANTUM PHASES FOR ELECTRIC CHARGES AND ELECTRIC (MAGNETIC) DIPOLES: PHYSICAL MEANING AND IMPLICATION

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We analyse the physical meaning of quantum phase effects for point-like charges and electric (magnetic) dipoles in an electromagnetic (EM) field. At present, there are known eight effects of such a kind: four of them (the magnetic and electric Aharonov – Bohm phases for electrons, the Aharonov – Casher phase for a moving magnetic dipole and the He – McKellar – Wilkens phase for a moving electric dipole) had been disclosed in 20th century, while four new quantum phases had recently been found by our team (A. L. Kholmetskii, O. V. Missevitch, T. Yarman). In our analysis of physical meaning of these phases, we adopt that a quantum phase for a dipole represents a superposition of quantum phases for each charge, composing the dipole. In this way, we demonstrate the failure of the Schrödinger equation for a charged particle in an EM field to describe new quantum phase effects, when the standard definition of the momentum operator is used. We further show that a consistent description of quantum phase effects for moving particles is achieved under appropriate re-definition of this operator, where the canonical momentum of particle in EM field is replaced by the interactional EM field momentum. Some implications of this result are discussed.

Keywords: quantum phase effects; electric dipole; magnetic dipole; Schrödinger equation; operator of momentum.

Introduction

As is well-known, at the middle of 20th century, Aharonov and Bohm predicted two quantum phase effects for electrons in an electromagnetic (EM) field [1; 2]: the electric Aharonov – Bohm (AB) effect with the phase

$$\delta_{\varphi} = \frac{e}{\hbar}\int \varphi dt, \tag{1}$$

and the magnetic Aharonov – Bohm (AB) effect with the phase

$$\delta_{A} = -\frac{e}{\hbar c}\int_{L} \boldsymbol{A}\cdot d\boldsymbol{s}. \tag{2}$$

Here φ, $\boldsymbol{A}$ are respectively scalar and vector potentials, $d\boldsymbol{s} = \boldsymbol{v}dt$ is the path element of a charge e along a line L, c is the light velocity in vacuum, and $\hbar$ is the reduced Planck constant.

Later, in the 1980s, the Aharonov – Casher (AC) phase for a moving magnetic dipole $\boldsymbol{m}$ in the presence of electric field $\boldsymbol{E}$ had been found [3]:

$$\delta_{mE} = \frac{1}{\hbar c}\int_{L} \left(\boldsymbol{m}\times\boldsymbol{E}\right)\cdot d\boldsymbol{s}, \tag{3}$$

and in the 1990s, the He – McKellar – Wilkens (HMW) phase for a magnetic dipole $\boldsymbol{p}$, moving in magnetic field $\boldsymbol{B}$, had been disclosed, too [4; 5]:

$$\delta_{pB} = -\frac{1}{\hbar c}\int_{L} \left(\boldsymbol{p}\times\boldsymbol{B}\right)\cdot d\boldsymbol{s}. \tag{4}$$

It is important to emphasize that the quantum phase effects (2)–(4) have been confirmed in corresponding experiments (e. g., [6–8]). The electric AB phase still was not directly observed, because the available attempts (see, e. g., [9]) were failed to reliably distinguish the electric AB phase (1) from the dynamical effects resulting due to non-vanishing electric component of the Lorentz force. Nevertheless, there are no doubts with respect to the reality of the electric AB phase (1), insofar as it directly derived, along with the magnetic AB phase (2), from the Schrödinger equation for a charged particle in an EM field, where the standard Hamiltonian

$$\hat{H} = \frac{\left(-i\hbar\nabla - \frac{e\boldsymbol{A}}{c}\right)^{2}}{2m} + e\varphi \tag{5}$$

is used (see, e. g., [10]).

A validity of this assertion follows from the common definition of quantum phase for charged particle in the presence of EM field,

$$\delta = \frac{1}{\hbar}\int \left(H - H_0\right) dt, \tag{6}$$

where H_0 stands for the Hamiltonian of particle in the absence of EM field. Therefore, combining equations (5) and (6), we obtain

$$\delta = \frac{1}{\hbar}\int e\varphi dt - \frac{1}{\hbar c}\int e\boldsymbol{A}\cdot d\boldsymbol{s}.$$

Thus, one can see that the first term on the right side describes the electric AB phase, while the second term corresponds to the magnetic AB phase.

Addressing now to equations (3), (4) for quantum phases for dipoles, we emphasize that both of them had been derived with some approximate Lagrangian expressions either for magnetic [3], or electric [4; 5] dipoles, which, in general, leave unanswered question about a possible existence of more quantum phase effects for moving dipoles.

We addressed the problem of deriving a covariant expression for the Lagrangian of electric/magnetic dipole, solving another task: to explore the origin of the high-temperature Kondo effect (the inverse dependence of the resistivity on the temperature for conducting materials [11]), which we experimentally revealed in iron-containing high-temperature superconductors [12; 13], and which supposedly could be explained via the interaction of magnetic dipole moments of conduction electrons with the magnetic dipole moments of impurities in superconductors [11]. Thus, in order to understand the manifestation of Kondo effects for our samples under investigation, we needed to use the force law for two interacting magnetic dipoles and, surprisingly, we did not find an unambiguous solution of this problem in the available literature.

Thus, in our own attempts to solve the problem, we obtained for the first time the relativistically invariant expression for the Lagrangian of electric (magnetic) dipole, and this result suggested us to look closer not only at the problem of high-temperature superconductivity, as exposed in references [12; 13], but also at the problem of quantum phase effects for moving dipoles and to derive the general expression for their total quantum phase (see section «Force law in material media via covariant Lagrangian and quantum phases for electric (magnetic) dipoles»).

In section «Physical meaning of quantum phases for electric (magnetic) dipoles and new quantum phases for point-like charges», we seek the physical meaning of quantum phases for dipoles as the superposition of quantum phases for electric charges composing the dipoles. In this way, we derive two novel quantum phase effects for point-like charges, which we named as the complementary magnetic and electric AB phases, correspondingly.

In section «Quantum phase effects for freely moving charges in an EM field and re-definition of the operator of momentum», we discuss the physical meaning of quantum phase effects for charges and dipoles, and show that their consistent description requires to re-define the momentum operator for charged particle via the sum of mechanical and EM momenta (41), instead of the old definition via the canonical momentum of the particle (36). In this way, we provide a clear physical interpretation of quantum phase effects for charged particle which indicates that even at its constant mechanical momentum (no force on the particle), the variation of its de Broglie wavelength happens entirely due to corresponding variation of interactional field momentum. Other important implications of the new definition of momentum operator (41) are also discussed.

Force law in material media via covariant Lagrangian and quantum phases for electric (magnetic) dipoles

During many decades, the problem of determination of a correct force law in material media was definitely underestimated by many researchers. In fact, it was tacitly supposed that the Lorentz force law, being well tested for point-like charged particles, can straightforwardly be extended to bound charges in material medium, which for a unit volume of such a medium takes the form (see, e. g., [14])

$$\boldsymbol{f}_L = \rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B}, \tag{7}$$

where $\boldsymbol{f}$ is the force density,

$$\rho = -\nabla\cdot\boldsymbol{P} \tag{8}$$

is the charge density, and

$$\boldsymbol{j} = \nabla\times\boldsymbol{M} + \frac{\partial\boldsymbol{P}}{\partial t} \tag{9}$$

is the current density; $\boldsymbol{P}$ being the polarization, and $\boldsymbol{M}$ being the magnetization.

However, equation (7) is, as minimum, incomplete, because it does not include the force density components, resulting from the secondary effects in material media, e. g.: its polarization, the emergence of mechanical stresses, etc.

In the second half of 20th century, it was proposed to describe the contribution of such secondary effects via introducing the hidden momentum of magnetic dipole (see, e. g., [15–17])

$$\boldsymbol{P}_h = \frac{1}{c}\boldsymbol{m} \times \boldsymbol{E}, \tag{10}$$

and the time variation of the density of hidden momentum (10) yields one more component of force, acting per unit volume of a magnetized material medium:

$$\boldsymbol{f}_h = \frac{1}{c}\frac{d}{dt}\left(\boldsymbol{M} \times \boldsymbol{E}\right). \tag{11}$$

Hence, it was adopted that the total force density on a material medium should be defined as the sum of the Lorentz components (7) and hidden momentum contribution (11).

However, we have shown in references [18; 19] that the sum $\boldsymbol{f}_L + \boldsymbol{f}_h$ is not Lorentz invariant, that leads to relativistically non-adequate results with respect to the force on electric (magnetic) dipoles as is seen by different inertial observers.

Thus, in order to find the correct relativistic expression for the force on a dipole, we suggested in references [18; 19] to apply an explicitly covariant expression for the Lagrangian density of a polarized or magnetized medium in an EM field [20]

$$l_{\text{int}} = \frac{1}{2}M^{\alpha\beta}F_{\alpha\beta}, \tag{12}$$

where $M^{\alpha\beta}$ is the magnetization-polarization tensor and $F^{\alpha\beta}$ is the tensor of EM field [14].

Integrating equation (12) over the volume of compact dipole, and introducing the rest mass M of the dipole, we obtain the total Lagrangian as follows:

$$L = -\frac{Mc^2}{\gamma} + \boldsymbol{p} \cdot \boldsymbol{E} + \boldsymbol{m} \cdot \boldsymbol{B}. \tag{13}$$

Substituting the Lagrangian (13) into the Euler – Lagrange equation at the given fields $\boldsymbol{E}$ and $\boldsymbol{B}$, i. e.,

$$\frac{\partial L}{\partial \boldsymbol{r}} = \frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}},$$

we arrive at the relativistic expression for the total force on the dipole [18; 19]

$$\boldsymbol{F} = \frac{d}{dt}\left(\gamma M\boldsymbol{v}\right) = \nabla\left(\boldsymbol{p} \cdot \boldsymbol{E}\right) + \nabla\left(\boldsymbol{m} \cdot \boldsymbol{B}\right) + \frac{d}{dt}\frac{\gamma\left(\boldsymbol{p}_{0\|} \cdot \boldsymbol{E}\right)\boldsymbol{v}}{c^2} +$$
$$+ \frac{d}{dt}\frac{\gamma\left(\boldsymbol{m}_{0\|} \cdot \boldsymbol{B}\right)\boldsymbol{v}}{c^2} + \frac{d}{dt}\frac{1}{c}\left(\boldsymbol{p}_0 \times \boldsymbol{B}\right) - \frac{d}{dt}\frac{1}{c}\left(\boldsymbol{m}_0 \times \boldsymbol{E}\right),$$

where the subscript 0 stands for the proper electric and magnetic dipole moments.

Next, we straightforwardly define the Hamilton function $H = \frac{\partial L}{\partial \boldsymbol{v}} \cdot \boldsymbol{v} - \boldsymbol{L}$, which, in the quantum limit, determines the Hamiltonian $\hat{H}$ and the total phase for a dipole in the presence of EM field [18; 19]:

$$\delta = \frac{1}{\hbar}\int \hat{H}dt = -\frac{1}{\hbar c^2}\int \gamma\left(\boldsymbol{p}_{0\|} \cdot \boldsymbol{E}\right)\boldsymbol{v} \cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int \gamma\left(\boldsymbol{m}_{0\|} \cdot \boldsymbol{B}\right)\boldsymbol{v} \cdot d\boldsymbol{s} +$$
$$+ \frac{1}{\hbar c}\int\left(\boldsymbol{m}_0 \times \boldsymbol{E}\right) \cdot d\boldsymbol{s} - \frac{1}{\hbar c}\int\left(\boldsymbol{p}_0 \times \boldsymbol{B}\right) \cdot d\boldsymbol{s} - \frac{1}{\hbar}\int\left(\boldsymbol{p} \cdot \boldsymbol{E}\right)dt - \frac{1}{\hbar}\int\left(\boldsymbol{m} \cdot \boldsymbol{B}\right)dt. \tag{14}$$

In what follows, we will consider the quantum phases for moving dipoles only, which allows us to exclude from further analysis the last two terms on the right side of equation (14), which respectively determine the Stark phase [21] and Zeeman phase [22], available for resting dipoles.

As is further shown in references [18; 19], in the weak relativistic limit, corresponding to the accuracy of calculations c^{-3}, the sum of remaining four terms in equation (14) can be written as

$$\delta \approx -\frac{1}{\hbar c}\int\left(\boldsymbol{p} \times \boldsymbol{B}\right) \cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int\left(\boldsymbol{p} \cdot \boldsymbol{E}\right)\boldsymbol{v} \cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int\left(\boldsymbol{m} \cdot \boldsymbol{B}\right)\boldsymbol{v} \cdot d\boldsymbol{s} + \frac{1}{\hbar c}\int\left(\boldsymbol{m} \times \boldsymbol{E}\right) \cdot d\boldsymbol{s}, \tag{15}$$

where all quantities are evaluated in a laboratory frame.

The first and the last terms on the right side of equation (15) stand respectively for the known HMW (4) and AC (3) phases, while the second and third terms correspond to the quantum phases previously unknown, which emerge under the motion of electric dipole in electric field

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int (\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s}, \tag{16}$$

and under the motion of magnetic dipole in magnetic field

$$\delta_{mB} = -\frac{1}{\hbar c^2}\int (\boldsymbol{m}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s}. \tag{17}$$

By such a way, we have obtained four quantum phases δ_{mE}, δ_{pB}, δ_{pE}, δ_{mB} for moving dipoles, defined by equations (3), (4), (16) and (17), correspondingly, and the subsequent problem is to disclose their physical meaning. We will solve this problem in section «Physical meaning of quantum phases for electric (magnetic) dipoles and new quantum phases for point-like charges» on the basis of a natural assumption that a quantum phase for a dipole represents a superposition of quantum phases for each charge, composing the dipole.

Physical meaning of quantum phases for electric (magnetic) dipoles and new quantum phases for point-like charges

In order to explore the relationship between quantum phases for charges and dipoles, we first of all point out that the fundamental AB phases in equations (1), (2) for point-like charges are defined via the EM field potentials and hence, we have to express the phases for dipoles (3), (4), (16), (17) through the EM field potentials, too.

Below, we solve this problem subsequently for each quantum phase of dipole. We will analyse the obtained expressions for the phases δ_{pB}, δ_{pE} with the standard model for electric dipole – two point-like charges $+e$ and $-e$, connected to each other via a rigid rod of a small distance d. Correspondingly, we will analyse the obtained expressions for the phases δ_{mE}, δ_{mB} using the standard model for magnetic dipole – an electrically neutral small conducting loop carrying a steady current with the density $\boldsymbol{j}$.

He – McKellar – Wilkens phase δ_{pB} (equation (4)). This phase depends on the magnetic field $\boldsymbol{B}$, and it can be expressed via the vector potential $\boldsymbol{A}$ in the following way:

$$\delta_{pB} = -\frac{1}{\hbar c}\int (\boldsymbol{p}\times\boldsymbol{B})\cdot d\boldsymbol{s} =$$
$$= -\frac{1}{\hbar c}\int_V\int_L (\boldsymbol{P}\times\boldsymbol{B})\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c}\int_V\int_L (\boldsymbol{P}\times(\nabla\times\boldsymbol{A}))\cdot d\boldsymbol{s}dV = \frac{1}{\hbar c}\int_V\int_L \rho\boldsymbol{A}\cdot d\boldsymbol{s}dV. \tag{18}$$

In the derivation of this equation, we have used the equality $\nabla\times\boldsymbol{A} = \boldsymbol{B}$, the definition $\boldsymbol{p} = \int_V \boldsymbol{P}dV$ (where $\boldsymbol{P}$ being the polarization, and V is the volume of the dipole), as well as the vector identity [23]

$$\int_S (\boldsymbol{A}\cdot\boldsymbol{P})d\boldsymbol{S} - \int_S \boldsymbol{P}(\boldsymbol{A}\cdot d\boldsymbol{S}) - \int_S \boldsymbol{A}(\boldsymbol{P}\cdot d\boldsymbol{S}) =$$
$$= \int_V \boldsymbol{A}\times(\nabla\times\boldsymbol{P})dV + \int_V \boldsymbol{P}\times(\nabla\times\boldsymbol{A})dV - \int_V \boldsymbol{A}(\nabla\cdot\boldsymbol{P})dV - \int_V \boldsymbol{P}(\nabla\cdot\boldsymbol{A})dV,$$

further on, we used the fact that the polarization $\boldsymbol{P}$ is vanishing on the surface S of dipole, so that all integrals on the left side are equal to zero. Finally, we have used equation (8) and the equality $\nabla\times\boldsymbol{P} = 0$ in the Coulomb gauge ($\nabla\cdot\boldsymbol{A} = 0$).

Thus, applying equation (18) to the model of electric dipole specified above, we obtain

$$\delta_{pB} = \frac{e}{\hbar c}\left(\oint_{L_+}\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{d})\cdot d\boldsymbol{s} - \oint_{L_-}\boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{s}\right), \tag{19}$$

where the path of the positive charge of dipole is designated as L_+, the path of the negative charge of dipole is designated as L_-, and $\boldsymbol{r}$ is the radial coordinate.

Equation (19) shows that the HMW phase δ_{pB} represents an algebraic sum of magnetic AB phases (2) for each charge, composing the dipole, and this result has already been derived in reference [24] soon after the discovery of the HMW phase.

At the same time, on can easily realize that the magnetic AB phase (2) cannot be responsible for other quantum phase effects derived for moving dipoles. What is more, one should notice that the second fundamental quantum phase for point-like charges – the electric AB phase (2) – does not explicitly contain the velocity of charge, and cannot be responsible for the origin of the remaining quantum phases δ_{mE}, δ_{pE}, δ_{mB}, for moving dipoles.

This circumstance makes rather interesting the problem of determination of their physical meaning, and next we consider one more quantum phase δ_{pE} for a moving electric dipole.

The phase δ_{pE} (equation (16)). This phase depends on the electric field $\boldsymbol{E}$, and we assume that it does not contain the inductive component (i. e., $\frac{\partial \boldsymbol{A}}{\partial t} = 0$). Then, using the equality $\boldsymbol{E} = -\nabla\varphi$, the definition $\boldsymbol{p} = \int_V \boldsymbol{P} dV$, as well as the equation (8), we obtain

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int(\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s} = \frac{1}{\hbar c^2}\int_V\oint(\boldsymbol{P}\cdot\nabla\varphi)\boldsymbol{v}\cdot d\boldsymbol{s}dV =$$

$$= \frac{1}{\hbar c^2}\int_V\oint\nabla\cdot(\boldsymbol{P}\varphi)\boldsymbol{v}\cdot d\boldsymbol{s}dV - \frac{1}{\hbar c^2}\int_V\oint\rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^2}\int_V\oint\rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV. \quad (20)$$

In the derivation of this equation, we also have used the vector identity $\nabla\cdot(\boldsymbol{P}\varphi) = \varphi\nabla\cdot\boldsymbol{P} + \boldsymbol{P}\cdot\nabla\varphi$, and have taken into account that the volume integral $\int_V\oint\nabla\cdot(\boldsymbol{P}\varphi)\boldsymbol{v}\cdot d\boldsymbol{s}dV$ can be transformed into a surface integral via the Gauss theorem, where the polarisation $\boldsymbol{P}$ is vanishing.

We further see that for the model of the electric dipole adopted above equation (20) takes the form

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int_V\int_L\rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV = -\frac{e}{\hbar c^2}\int_{L_+}\varphi(\boldsymbol{r}+\boldsymbol{d})\boldsymbol{v}\cdot d\boldsymbol{s} + \frac{e}{\hbar c^2}\int_{L_+}\varphi(\boldsymbol{r})\boldsymbol{v}\cdot d\boldsymbol{s}.$$

This equation indicates that the quantum phase δ_{pE} for electric dipole represents a superposition of new quantum phases for point-like charges composing the dipole, which is defined by the equality

$$\delta_{c\varphi} = -\frac{e}{\hbar c^2}\int\varphi\boldsymbol{v}\cdot d\boldsymbol{s}. \quad (21)$$

One can see that in the weak relativistic limit, the phase (21) is smaller than the electric AB phase (1) by $\left(\frac{v}{c}\right)^2$ times, and in references [25; 26] we named it as a complementary electric AB phase, supplying it with the subscript c.

Aharonov – Casher phase δ_{mE} (equation (3)). This phase depends on the electric field $\boldsymbol{E}$, and we again assume that it does not contain the inductive component. Then, using the equality $\boldsymbol{E} = -\nabla\varphi$, as well as the definition $\boldsymbol{m} = \int\boldsymbol{M}dV$, we obtain

$$\delta_{mE} = -\frac{1}{\hbar c}\int_V\int(\boldsymbol{M}\times\nabla\varphi)\cdot d\boldsymbol{s}dV =$$

$$= \frac{1}{\hbar c}\int_V\int\nabla\times(\boldsymbol{M}\varphi)\cdot d\boldsymbol{s}dV - \frac{1}{\hbar c}\int_V\int\varphi(\nabla\times\boldsymbol{M})\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c}\int_V\oint\varphi(\nabla\times\boldsymbol{M})\cdot d\boldsymbol{s}dV. \quad (22)$$

Here, we have used the vector identity $\nabla\times(\boldsymbol{M}\varphi) = \varphi\nabla\times\boldsymbol{M} - \boldsymbol{M}\times\nabla\varphi$ and taken into account that the volume integral $\int_V\oint\nabla\times(\boldsymbol{M}\varphi)\cdot d\boldsymbol{s}dV$ can be transformed into a surface integral, where the magnetisation $\boldsymbol{M}$ is vanishing.

For further transformation of integral (22), we apply the equality (9) and assume a stationary polarisation, where $\frac{\partial\boldsymbol{P}}{\partial t} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{P}$. Hence, we derive

$$\delta_{mE} = -\frac{1}{\hbar c^2}\int_V\oint\varphi(\boldsymbol{j} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{P})\cdot d\boldsymbol{s}dV. \quad (23)$$

For the adopted model of magnetic dipole – an electrically neutral current loop with a steady current – the proper polarisation is equal to zero. Therefore, the polarisation $\boldsymbol{P}$ in equation (23) can emerge for a moving dipole only, and equal to $\boldsymbol{P} = \boldsymbol{v} \times \frac{\boldsymbol{M}_0}{c}$. Hence, one sees that it is orthogonal to the vector $d\boldsymbol{s}$, and equation (23) yields

$$\delta_{mE} = -\frac{1}{\hbar c^2}\int_V \oint \varphi \boldsymbol{j} \cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho \boldsymbol{u} \cdot d\boldsymbol{s}dV, \tag{24}$$

where we have used the equality $\boldsymbol{j} = \rho\boldsymbol{u}$.

Thus, the obtained expression (24) for the AC phase via the scalar potential φ indicates that this phase represents the superposition of complementary electric AB phases (21) for all charged, composing the magnetic dipole.

More specifically, the positive charges of the frame of the loop (which rest in its proper frame), contribute to the total phase for a moving magnetic dipole at the value

$$\left(\delta_{mE}\right)_{\text{positive}} = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho_+ \boldsymbol{v} \cdot d\boldsymbol{s}dV, \tag{25}$$

whereas the negative charged (the carries of current) contribute at the value

$$\left(\delta_{mE}\right)_{\text{negative}} = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho_- \left(\boldsymbol{u} + \boldsymbol{v}\right) \cdot d\boldsymbol{s}dV. \tag{26}$$

Summing up equations (25) and (26) at $\rho_+ = -\rho_-$, we arrive at the phase (24).

Thus, we reveal that the AC phase for a moving magnetic dipole (3) represents the superposition of complementary electric AB phases (21) for all charges of the dipole.

The phase δ_{mB} (equation (17)). In order to express this phase via the vector potential $\boldsymbol{A}$, we will use the equality $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ along with the definition $\boldsymbol{m} = \int \boldsymbol{M}dV$ and the vector identity $\nabla \cdot \left(\boldsymbol{M} \times \boldsymbol{A}\right) = \boldsymbol{M} \cdot \left(\nabla \times \boldsymbol{A}\right) + \boldsymbol{A} \cdot \left(\nabla \times \boldsymbol{M}\right)$. Hence, we derive from equation (17):

$$\delta_{mB} = -\frac{1}{\hbar c^2}\int_V \oint \left(\boldsymbol{M} \cdot \left(\nabla \times \boldsymbol{A}\right)\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV = \frac{1}{\hbar c^2}\int_V \oint \left(\nabla \cdot \left(\boldsymbol{M} \times \boldsymbol{A}\right)\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV -$$
$$- \frac{1}{\hbar c^2}\int_V \oint \left(\boldsymbol{A} \cdot \left(\nabla \times \boldsymbol{M}\right)\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^2}\int_V \oint \left(\boldsymbol{A} \cdot \left(\nabla \times \boldsymbol{M}\right)\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV, \tag{27}$$

where we have taken into account that the first integral on the right side of equation (27) is vanishing due to the Gauss theorem. In order to evaluate the remaining integral, we involve the equality (9) and assume again that the polarisation is stationary, i. e. $\frac{d\boldsymbol{P}}{dt} = \frac{\partial \boldsymbol{P}}{\partial t} + \left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{P} = 0$, and $\frac{\partial \boldsymbol{P}}{\partial t} = -\left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{P}$. Hence, combining equations (9) and (27), one gets

$$\delta_{mB} = -\frac{1}{\hbar c^3}\int_V \int \left(\boldsymbol{A} \cdot \boldsymbol{j} + \boldsymbol{A} \cdot \left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{P}\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV. \tag{28}$$

The second integral in equation (28) can be transformed to the form:

$$\int_V \int \left(\boldsymbol{A} \cdot \left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{P}\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV = \int_V \int \left(\boldsymbol{A} \cdot \left(\boldsymbol{v} \cdot \nabla\right)\boldsymbol{P}\right)v^2 dtdV = v^2 \int_V \int \frac{d}{dt}\left(\boldsymbol{A} \cdot \boldsymbol{P}\right)dtdV - v^2 \int \left(\boldsymbol{p} \cdot \frac{d\boldsymbol{A}}{dt}\right)dt.$$

Therefore, it is vanishing under adoption of the natural conditions $\boldsymbol{A}\left(t = 0\right) = \boldsymbol{A}\left(t = \infty\right) = 0$. Hence, equation (28) yields

$$\delta_{mB} = -\frac{1}{\hbar c^3}\int_V \int \left(\boldsymbol{j} \cdot \boldsymbol{A}\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^3}\int_V \int \left(\rho\boldsymbol{u} \cdot \boldsymbol{A}\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV, \tag{29}$$

where we have used the equality $\boldsymbol{j} = \rho\boldsymbol{u}$.

Applying equation (29) to the adopted model of a magnetic dipole, we can write by analogy with equations (25) and (26):

$$\left(\delta_{mE}\right)_{\text{positive}} = -\frac{1}{\hbar c^2}\int_V \oint \rho_+ \left(\boldsymbol{v} \cdot \boldsymbol{A}\right)\boldsymbol{v} \cdot d\boldsymbol{s}dV,$$

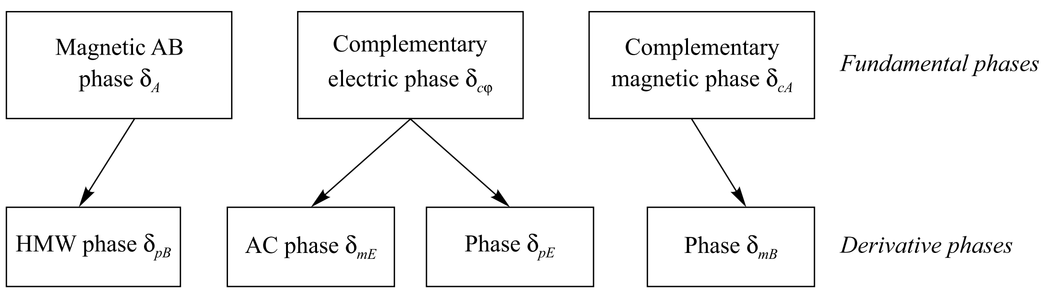
$$\left(\delta_{mE}\right)_{\text{negative}} = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho_- \left(\left(\boldsymbol{u} + \boldsymbol{v}\right) \cdot \boldsymbol{A}\right)\left(\boldsymbol{u} + \boldsymbol{v}\right) \cdot d\boldsymbol{s}dV.$$

These equations show that the phase δ_{mB} for a moving magnetic dipole represents a superposition of new quantum phases for point-like charges of the dipole, being defined by the equality

$$\delta_{cA} = -\frac{e}{\hbar c^3}\int (\boldsymbol{v}\cdot\boldsymbol{A})\boldsymbol{v}\cdot d\boldsymbol{s}, \tag{30}$$

which we named as the complementary magnetic AB phase [25; 26]. It is seen that this phase is approximately by $\left(\frac{v}{c}\right)^2$ times smaller than the magnetic AB phase (2).

Thus, all quantum phases δ_{mE}, δ_{pB}, δ_{pE}, δ_{mB} for an electric or magnetic dipole moving in EM field originate from three fundamental phases for point-like charges: the magnetic AB phase δ_A (2), the complementary electric AB phase $\delta_{c\varphi}$ (21), as well as the complementary magnetic AB phase δ_{cA} (30). A relationship between fundamental velocity-dependent quantum phase effects for point-like charges and phase effects for electric (magnetic) dipoles is shown in figure, which illustrates their physical meaning.



Relationship between velocity-dependent quantum phases for charged particles and for moving dipoles

We found above that the ratio of complementary electric AB phase (21) and electric AB phase (1) has the order of $\left(\frac{v}{c}\right)^2$, and the same result is valid with respect to the ratio of complementary magnetic AB phase (30) and magnetic AB phase (2). This observation led us to assume in reference [25] that the phases $\delta_{c\varphi}$, δ_{cA} could represent some relativistic extension of the fundamental electric and magnetic AB phases, correspondingly.

However, later we pointed out [26] that the Schrödinger equation for charged particle in an EM field does already contain the terms of order c^{-2}. Nevertheless, it does not include the complementary electric AB phase $\delta_{c\varphi}$ of the same order. This already signifies that the actual situation with respect to physical interpretation of the new quantum phases $\delta_{c\varphi}$, δ_{cA} is more complicated, and this problem is analysed in the next section.

Quantum phase effects for freely moving charges in an EM field and re-definition of the operator of momentum

The results, which we obtained above, indicate that a moving point-like charge in the presence of EM field is characterised by three quantum phases: the previously known magnetic AB phase (2), as well as the complementary electric (21) and magnetic (30) AB phases, disclosed via the analysis of quantum phases for moving dipoles [18; 19; 25; 26]. Therefore, the determination of physical meaning of these phase effects acquires the fundamental importance.

Analysing this problem, we assume that all quantum phase effects for a moving charge should be directly related to its wave vector $\boldsymbol{k}$ and de Broglie wavelength λ, which can depend not only on the mechanical momentum of the particle $\boldsymbol{P}_M$, but also on the EM momentum $\boldsymbol{P}_{EM}$ for a system «charged particle plus external EM field». This assumption suggests us to generalise the wave vector in the form

$$\boldsymbol{k} = \frac{(\boldsymbol{P}_M + \boldsymbol{P}_{EM})}{\hbar}, \tag{31}$$

with the corresponding de Broglie wavelength

$$\lambda = \frac{h}{|\boldsymbol{P}_M + \boldsymbol{P}_{EM}|}. \tag{32}$$

Hence, the corresponding quantum phase $d\delta$ along the path $d\boldsymbol{s}$ is equal to

$$d\delta = -\boldsymbol{k}\cdot d\boldsymbol{s} = -\frac{\left(\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right)\cdot d\boldsymbol{s}}{\hbar}. \tag{33}$$

In order to verify this assumption, we address to reference [26], where we calculated the interaction EM field momentum for a spinless charged particle in the external $\boldsymbol{E}$, $\boldsymbol{B}$ fields as the function of scalar φ and vector $\boldsymbol{A}$ potentials:

$$\boldsymbol{P}_{EM} = \frac{1}{4\pi c}\int_V \left(\boldsymbol{E}\times\boldsymbol{B}_e\right)dV + \frac{1}{4\pi c}\int_V \left(\boldsymbol{E}_e\times\boldsymbol{B}\right)dV = \frac{e\boldsymbol{A}}{c} + \frac{\boldsymbol{v}e\varphi}{c^2} + \frac{e\boldsymbol{v}\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)}{c^3}, \tag{34}$$

here $\boldsymbol{E}_e$, $\boldsymbol{B}_e$ are the electric and magnetic fields of a moving charge.

Thus, combining equations (33), (34), and integrating over the path $\boldsymbol{s}$, we obtain the total phase of charged particle, moving in EM field:

$$\delta = -\boldsymbol{k}\cdot d\boldsymbol{s} = -\frac{1}{\hbar}\int \boldsymbol{P}_M\cdot d\boldsymbol{s} - \frac{e}{\hbar c}\int \boldsymbol{A}\cdot d\boldsymbol{s} - \frac{e}{\hbar c^2}\int \varphi\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{e}{\hbar c^3}\int \boldsymbol{v}\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)\cdot d\boldsymbol{s}. \tag{35}$$

The first term on the right side of this equation describes the phase of particle, associated with its mechanical momentum, which exists in the absence of EM field, while the remaining three terms stand for the magnetic AB phase (2), the complementary electric AB phase (21) and complementary magnetic AB phase (30), correspondingly. This result fully validates our assumptions ((31), (32)) with respect to the dependence of the wave vector $\boldsymbol{k}$ and the de Broglie wavelength λ on the scalar and vector potentials.

At the same time, one should notice that in no way the phase (35) can be derived via the Schrödinger equation with the standard Hamiltonian (5) for charged particle in an EM field. As we have already shown in the introductory section, the Hamiltonian (5), being substituted into equation (6) for the total phase of particle, yields only the magnetic AB effect for velocity-dependent phase, leaving non-accounted the complementary magnetic and electric AB phases.

This contradictory situation, disclosed at the first time in reference [26], suggested us to look closer at the adopted procedure of transition from the classical to quantum description of charged particles. On this way, we concluded [26; 27] that the failure of the Hamiltonian (5) to describe the complementary electric and complementary magnetic AB phases, playing important role in the physical interpretation of a full set of quantum phases for charges and dipoles, definitely indicates the presence of a fundamental inconsistency in quantum description of charged particles in an EM field.

According to our analysis [26; 27], such an inconsistency is present in the definition of the momentum operator for charged particle in EM field, which is commonly associated with its canonical momentum, i. e.,

$$\boldsymbol{P}_c = \boldsymbol{P}_M + \frac{e\boldsymbol{A}}{c} \to \hat{\boldsymbol{P}}_c = -i\hbar\nabla. \tag{36}$$

Hence, equation (36) straightforwardly yields the Hamiltonian (5).

In fact, the postulate (36) tacitly prescribes the fundamental role to the canonical momentum of charged particle in quantum mechanics, which, however, looks not so obvious. Thus, one can wonder, why the physical context of equation (36) was, to the best of our knowledge, not discussed earlier, before our publications [26; 27].

As is known, the canonical momentum (36) for a charged particle in an EM field emerges as a formal variable in the Euler – Lagrange equation of classical electrodynamics (see, e. g., [28]), and a question about a physical meaning of $\boldsymbol{P}_c$ even was not discussed.

We separately investigated this problem in reference [26] and found that the term $\frac{e\boldsymbol{A}}{c}$ describes the interaction EM field momentum for the system «charged particle in an external electric and magnetic fields» in the particular case of zero velocity of particle.

Indeed, for a charged particle at rest, its magnetic field is equal to zero, so that the interactional field momentum takes the form

$$\boldsymbol{P}_{EM}\left(v=0\right) = \frac{1}{4\pi c}\int_V \left(\boldsymbol{E}_e\times\boldsymbol{B}\right)dV = \frac{1}{4\pi c}\int_V \left(\boldsymbol{E}_e\times\left(\nabla\times\boldsymbol{A}\right)\right)dV, \tag{37}$$

where we have used the equality $\boldsymbol{B} = \left(\nabla\times\boldsymbol{A}\right)$. Further, we involve the vector identity [23]

$$\int_V \left(\boldsymbol{E}_e\times\left(\nabla\times\boldsymbol{A}\right)\right)dV + \int_V \left(\boldsymbol{A}\times\left(\nabla\times\boldsymbol{E}_e\right)\right)dV - \int_V \left(\boldsymbol{E}_e\left(\nabla\cdot\boldsymbol{A}\right)\right)dV - \int_V \left(\boldsymbol{A}\left(\nabla\cdot\boldsymbol{E}_e\right)\right)dV = 0,$$

which in the Coulomb gauge $(\nabla\cdot\boldsymbol{A} = 0)$ yields

$$\int_V \left(\boldsymbol{E}_e \times \left(\nabla \times \boldsymbol{A}\right)\right) dV = -\int_V \left(\boldsymbol{A} \times \left(\nabla \times \boldsymbol{E}_e\right)\right) dV + 4\pi \int_V \left(\rho_e \boldsymbol{A}\right) dV = \frac{e\boldsymbol{A}}{c}. \tag{38}$$

Here we have taken into account that $\nabla \times \boldsymbol{E}_e = 0$ for a resting particle, and used the Maxwell equation $\nabla \cdot \boldsymbol{E}_e = 4\pi\rho_e$, ρ_e being the charge density of the particle. Hence, using equations (37) and (38), we obtain

$$\boldsymbol{P}_{EM}\left(v = 0\right) = \frac{e\boldsymbol{A}}{c}. \tag{39}$$

Further on, combining equations (36) and (39), we arrive at the equality

$$\boldsymbol{P}_c = \boldsymbol{P}_M\left(\boldsymbol{v}\right) + \boldsymbol{P}_{EM}\left(v = 0\right) \rightarrow \hat{\boldsymbol{P}} = -i\hbar\nabla, \tag{40}$$

which shows that the canonical momentum represents the sum of mechanical momentum of *moving* particle $\boldsymbol{P}_M$ and the interactional EM field momentum $\boldsymbol{P}_{EM}$ in the situation, where the particle *would be at rest* in the frame of observation. Thus, equation (40) indicates that the canonical momentum $\boldsymbol{P}_c$ does not have a real physical meaning.

Under this circumstance, it seems attractive to re-define the momentum operator in such a way, where the sum of mechanical momentum and interactional EM field momentum are taken at the same velocity $\boldsymbol{v}$ of the charge, i. e.,

$$\boldsymbol{P}_M\left(\boldsymbol{v}\right) + \boldsymbol{P}_{EM}\left(\boldsymbol{v}\right) \rightarrow \hat{\boldsymbol{P}} = -i\hbar\nabla. \tag{41}$$

Hence, instead of the Hamiltonian (5), we get

$$\hat{H} = \frac{\left(-i\hbar\nabla - \boldsymbol{P}_{EM}\right)^2}{2M} + e\varphi,$$

or, in the explicit form (see equation (34)),

$$H = \frac{1}{2M}\left(\boldsymbol{P} - \frac{e\boldsymbol{A}}{c} - \frac{\boldsymbol{v}e\varphi}{c^2} - \frac{e\boldsymbol{v}\left(\boldsymbol{A} \cdot \boldsymbol{v}\right)}{c^3}\right)^2 + e\varphi, \tag{42}$$

where all variables are considered as operators.

Presenting in equation (42) $\boldsymbol{P} = M\boldsymbol{v}$ and assuming the Coulomb gauge, where the operators $\boldsymbol{v}$ and $\boldsymbol{A}$ commutate with each other, we derive to the accuracy of calculations c^{-3}:

$$H = -\frac{\hbar^2}{2M}\Delta + e\varphi - \frac{e\boldsymbol{A} \cdot \boldsymbol{v}}{c} - \frac{e\varphi v^2}{c^2} - \frac{ev^2\left(\boldsymbol{A} \cdot \boldsymbol{v}\right)}{c^3}, \tag{43}$$

where we neglected the term $\frac{e^2 A^2}{2Mc^2}$ in comparison with other terms of equation (43), which in any practical situation is quite warranted, and Δ stands for the Laplacian.

Thus, substituting the Hamiltonian (43) into equation (6), we derive the quantum phase for a moving charged particle in the presence of EM field as

$$\delta = \frac{e}{\hbar}\int \varphi dt - \frac{e}{\hbar c}\int \boldsymbol{A} \cdot d\boldsymbol{s} - \frac{e}{\hbar c^2}\int \varphi \boldsymbol{v} \cdot d\boldsymbol{s} - \frac{e}{\hbar c^3}\int \boldsymbol{v}\left(\boldsymbol{A} \cdot \boldsymbol{v}\right) \cdot d\boldsymbol{s}, \tag{44}$$

where the first two terms on the right side stand for the electric (1) and magnetic (2) AB phases, while the third and fourth terms describe the complementary electric (21) and magnetic (30) AB phases, corresponsingly, which we disclosed in references [25; 26].

Thus, the re-definition of the momentum operator (41), which we suggested instead of the customary definition (36), actually allows describing all quantum phase effects for charged particles in EM field, and to ensure a full harmony between equations (44) and (15), describing quantum phase effects respectively for charges and dipoles, with their relationship according to figure.

We add that the negative sign for the velocity-dependent quantum phases in equation (44) reflects the inverse dependence of the de Broglie wavelength of charged particle λ on the interactional EM field momentum according to equation (32).

Conclusion

Thus, analysing quantum phase effects for moving dipoles, we have shown that, in addition to the known Aharonov – Casher and He – McKellar – Wilkens phases (equations (3) and (4), correspondingly), there are two novel phases (16), (17), and the entire set of the phases δ_{mE}, δ_{pB}, δ_{pE}, δ_{mE}, correspond to a full set of combinations between the pair $\boldsymbol{p}$, $\boldsymbol{m}$ and the pair $\boldsymbol{E}$, $\boldsymbol{B}$.

These results made topical the problem of determination of the physical meaning for quantum phases of moving dipoles, and in references [25; 26] we suggested to explain their origin via the superposition of fundamental quantum phases for point-like charges composing the dipoles.

On this way, we discovered two novel fundamental phase effects with the complementary electric $\delta_{c\varphi}$ (21) and complementary magnetic δ_{cA} (30) AB phases, correspondingly.

The disclosure of all quantum phase effects for point-like charges allowed us to conclude that the de Broglie wavelength for a moving charged particle depends not only on its mechanical momentum $\boldsymbol{P}_M$, but also on the interactional EM momentum $\boldsymbol{P}_M$ via the modulus of the vector sum of mechanical and EM momenta $|\boldsymbol{P}_M + \boldsymbol{P}_{EM}|$ (see equation (32)). The latter equation allows us to understand all quantum phase effects for a moving charge as a corresponding variation of its de Broglie wavelength with the scalar and vector potentials.

One should further notice that the direct observation of the phases $\delta_{c\varphi}$, δ_{cA} is hardly possible for non-relativistic charges, where the electric (δ_{φ}) and magnetic (δ_A) AB phases strongly dominate. However, for electrically neutral dipoles, the phases δ_{φ}, δ_A are vanishing, that opens the principal possibility to measure, at least indirectly, the complementary phases $\delta_{c\varphi}$, δ_{cA}.

In particular, as we have shown above, the AC phase for a moving magnetic dipole in an electric field represents the superposition of $\delta_{c\varphi}$ phases for all charges of the dipole and hence, the experimental observation of the AC phase [7] does prove the existence of complementary electric AB phase $\delta_{c\varphi}$, too. This result already indicates the need to re-define the momentum operator according to equation (41), where it is associated with the sum of mechanical momentum and EM momentum for a charged particle in an EM field. Then, as we have shown above, the Schrödinger equation for a charged particle with the momentum operator (41) yields equation (44) for the total phase of such particle, which contains both the previously known AB electric and magnetic phases (the first and second terms on the right side), as well as previously unknown complementary electric and magnetic AB phases (the third and fourth terms on the right side).

Further, we emphasise that the proposed re-definition of the momentum operator (41) must be universal, and also applicable to the Klein – Gordon equation and the Dirac equation. In this respect we remind that known fact that the sum of mechanical momentum of particle $\boldsymbol{P}_M$ and interactional field momentum $\boldsymbol{P}_{EM}$ represents the spatial components of the four-vector, whose time component is defined as the sum of the energy of particle and the energy of interactional EM field. Thus, the proposed re-definition of the operator of momentum (41) keeps the Lorentz invariance of the Klein – Gordon and Dirac equations, and allows us to describe the fundamental phase effects (44) for freely moving charge, too.

With respect to electrically bound quantum systems, in reference [27] we suggested the corresponding modification of fundamental equations of atomic physics with the suggested re-definition of the momentum operator (41), and have shown that this way promises the elimination of the available subtle deviations between calculated and measured data in precise physics of simple atoms.

However, this problem lies outside the scope of the present paper, and will be analysed elsewhere.

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